

Comparison Theorems for Backward Stochastic Volterra Integral Equations*

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Abstract

For backward stochastic Volterra integral equations (BSVIEs) in multi-dimensional Euclidean spaces, comparison theorems are established in a systematic way for the adapted solutions and adapted M-solutions. For completeness, comparison theorems for (forward) stochastic differential equations, backward stochastic differential equations, and (forward) stochastic Volterra integral equations (FSVIEs) are also presented. Duality principles are used in some relevant proofs. Also, it is found that certain kind of monotonicity conditions play crucial roles to guarantee the comparison theorems for FSVIEs and BSVIEs to be true. Various counterexamples show that the assumed conditions are almost necessary in some sense.

Keywords. Forward stochastic Volterra integral equations, backward stochastic Volterra integral equation, comparison theorem, duality principle.

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1 Introduction.

Throughout this paper, we let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $W(\cdot)$ is defined with $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ being its natural filtration augmented by all the \mathbb{P} -null sets. We consider the following equation in \mathbb{R}^n , the usual n -dimensional real Euclidean space:

$$Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t))ds - \int_t^T Z(t, s)dW(s), \quad t \in [0, T], \quad (1.1)$$

which is called a *backward stochastic Volterra integral equation* (BSVIE, for short). Such kind of equations have been investigated in the recent years (see [15, 23, 24, 25, 21, 2] and references cited therein). BSVIEs are natural extensions of by now well-understood backward stochastic differential equations (BSDEs, for short) whose integral form is as follows:

$$Y(t) = \xi + \int_t^T g(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \quad t \in [0, T]. \quad (1.2)$$

See [18, 10, 16, 27] for some standard results on BSDEs. An interesting result of BSDEs is the comparison theorem for the adapted solutions. A little precisely, say, for $n = 1$, if $(Y^i(\cdot), Z^i(\cdot))$ is the adapted solution

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to the BSDE (1.2) with $(\xi, g(\cdot))$ replaced by $(\xi^i, g^i(\cdot))$ ($i = 0, 1$) such that

$$\begin{cases} \xi^0 \leq \xi^1, & \text{a.s.}, \\ g^0(t, y, z) \leq g^1(t, y, z), & \forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}, \text{ a.s.}, \end{cases} \quad (1.3)$$

then

$$Y^0(t) \leq Y^1(t), \quad t \in [0, T], \text{ a.s.} \quad (1.4)$$

The comparison theorem also holds for multi-dimensional BSDEs. We refer the readers to [14] for details. Because of the comparison theorem, one can use the adapted solutions to BSDEs as dynamic risk measures or stochastic differential utility for (static) random variables which could be the payoff of a European contingent claim at the maturity.

Now, for BSVIEs, from mathematical point of view, it is natural to ask if a comparison theorem similar to that for BSDEs hold for solutions to BSVIEs. More precisely, if $(Y^i(\cdot), Z^i(\cdot, \cdot))$ is the solution to BSVIE (1.1), in a proper sense, with $(\psi(\cdot), g(\cdot))$ replaced by $(\psi^i(\cdot), g^i(\cdot))$, $i = 0, 1$, and

$$\begin{cases} \psi^0(t) \leq \psi^1(t), & t \in [0, T], \text{ a.s.}, \\ g^0(t, s, y, z, \zeta) \leq g^1(t, s, y, z, \zeta), & 0 \leq t \leq s \leq T, y, z, \zeta \in \mathbb{R}, \text{ a.s.} \end{cases} \quad (1.5)$$

Can we have the comparison relation (1.4)?

On the other hand, similar to BSDEs, if proper comparison theorems hold for BSVIEs, then there will be some interesting applications of BSVIEs in risk management and optimal investment/consumption problems. Let us elaborate in a little details.

It is common that in order to expect some returns from various existing risky assets, one should hold them for possibly different length of time period. The value of the positions for these assets at some future time form a (not necessarily adapted) stochastic process, for which people would like to measure the dynamic risks. A simple illustrative example can be found in [24]. We emphasize that the processes (not just random variables) for which one wants to measure the dynamic risk are not necessarily adapted. Dynamic risk measures for discrete-time processes have been considered in the literature, see, for examples, [12, 7, 1] and so on. On the other hand, static risk measures for continuous-time processes were studied in [5, 6]. We believe that BSVIEs should be a useful tool in studying dynamic risk measures for (not necessarily adapted) stochastic processes. Therefore, to establish comparison theorems for BSVIEs becomes quite necessary.

The second relevant motivation comes from the study of general yet realistic stochastic utility problem. The stochastic differential utility was introduced and studied in [8, 11], where the intertemporal consistency and Bellman's principle of optimality is applicable. However, real problems are usually of time-inconsistent nature. In fact, many experimental study on time preference shows that the standard assumption of time consistency is unrealistic. Moreover, substantial evidence also suggest that agents are impatient about choices in the short term but are patient among the long-term alternatives. Recently, some people are interested in the following type of stochastic utility function

$$Y(t) = \mathbb{E} \left[\int_t^T \ell(t, s) u(c(s)) ds \middle| \mathcal{F}_t \right], \quad t \in [0, T],$$

with $\ell(t, s)$ being the discount factor, see [17, 9, 26]. We expect that comparison theorems of BSVIEs will play an important role in formulating general stochastic utility functions and investigating their properties such as comparative risk aversion, risk aversion, etc., which will substantially extend the results in [8].

We will present applications of comparison theorems of BSVIEs in finance and other related area in our future publications.

Now, returning to comparison theorems for BSVIEs, we point out that unlike BSDEs, (1.5) is not enough to ensure comparison relation (1.4), in general. Various counterexamples will be presented. Due to the complicated situation for BSVIEs, the theory of comparison for solutions to BSVIEs is much more richer than that for BSDEs. The main purpose of this paper is to establish various comparison theorems for solutions to BSVIEs in multi-dimensional Euclidean spaces. To this end, we first will consider BSVIE (1.1) with the generator $g(\cdot)$ independent of $Z(s, t)$. For such a case, in order the comparison theorem holds, one needs some kind of monotonicity for the generator $g(\cdot)$ and/or the free term $\psi(\cdot)$. Some examples will show that the conditions we impose are almost necessary. The second case to be considered is that the generator $g(\cdot)$ depends on $Z(s, t)$ and independent of $Z(t, s)$. For such a case, we are comparing adapted M-solution for (1.1) introduced in [25]. It turns out that under proper monotonicity conditions, we are able to obtain a comparison theorem for adapted M-solutions, which is weaker than that for the first case. More precisely, instead of (1.4), we can only have

$$\mathbb{E}\left[\int_t^T Y^0(s)ds|\mathcal{F}_t\right] \leq \mathbb{E}\left[\int_t^T Y^1(s)ds|\mathcal{F}_t\right], \quad t \in [0, T], \text{ a.s.}$$

This result corrects a relevant result in [23, 24]. Finally, inspired by [4] and [3], we introduce a new notion, called *conditional h-solutions* for BSVIEs (1.1), and briefly discuss the corresponding comparison theorem by following similar ideas for the first two cases.

Note that the proofs of above results are closely connected with the comparison theorems of (forward) stochastic differential equations (FSDEs, for short), (forward) stochastic Volterra integral equations (FSVIEs, for short), and BSDEs (allowing the dimension $n > 1$). For completeness, we will present/recall some relevant results here. Interestingly, even for FSDEs and BSDEs, our proofs are different from those in [13, 19, 14], respectively, and more straightforward.

The rest of the paper is organized as follows: In Section 2, we present some comparison theorems of FSDEs, BSDEs and FSVIEs. In Section 3, we establish several comparison theorems for BSVIEs from three different perspectives. Various persuasive examples will be presented to illustrate the obtained results. Finally, some concluding remarks are collected in Section 4.

2 Comparison theorem for FSDEs, FSVIEs, and BSDEs

In this section, we are going to present comparison theorems for FSDEs, FSVIEs, and BSDEs, allowing the dimension $n > 1$. Some of them are known. But our proofs are a little different.

Let us first make some preliminaries. Denote

$$\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, 1 \leq i \leq n\}.$$

When $x \in \mathbb{R}_+^n$, we also denote it by $x \geq 0$, and say that x is *nonnegative*. By $x \leq 0$ and $x \geq y$ (for $x, y \in \mathbb{R}^n$), we mean $-x \geq 0$ and $x - y \geq 0$, respectively. In what follows, we let $e_i \in \mathbb{R}_+^n$ be the vector that the i -th entry is 1 and all other entries are zero. Let

$$\begin{cases} \mathbb{R}_+^{n \times m} = \{A = (a_{ij}) \in \mathbb{R}^{n \times m} \mid a_{ij} \geq 0, 1 \leq i \leq n, 1 \leq j \leq m\}, \\ \mathbb{R}_{*+}^{n \times n} = \{A = (a_{ij}) \in \mathbb{R}^{n \times n} \mid a_{ij} \geq 0, i \neq j\} \equiv \{A \in \mathbb{R}^{n \times n} \mid \langle Ae_i, e_j \rangle \geq 0, i \neq j\}, \\ \mathbb{R}_d^{n \times n} = \{A = (a_{ij}) \in \mathbb{R}^{n \times n} \mid a_{ij} = 0, i \neq j\} \equiv \{A \in \mathbb{R}^{n \times n} \mid \langle Ae_i, e_j \rangle = 0, i \neq j\}. \end{cases}$$

Note that $\mathbb{R}_+^{n \times m}$ is the set of all $(n \times m)$ matrices with all the entries being nonnegative, $\mathbb{R}_{*+}^{n \times n}$ is the set of all $(n \times n)$ matrices with all the off-diagonal entries being nonnegative (no conditions are imposed on the diagonal entries), and $\mathbb{R}_d^{n \times n}$ is the set of all $(n \times n)$ diagonal matrices, with the diagonal entries allowing

to be any real numbers. Clearly, $\mathbb{R}_+^{n \times m}$ and $\mathbb{R}_{*+}^{n \times n}$ are closed convex cones of $\mathbb{R}^{n \times m}$ and $\mathbb{R}^{n \times n}$, respectively; $\mathbb{R}_+^{n \times n}$ is a proper subset of $\mathbb{R}_{*+}^{n \times n}$; and $\mathbb{R}_d^{n \times n}$ is a proper subspace of $\mathbb{R}^{n \times n}$, contained in $\mathbb{R}_{*+}^{n \times n}$. Also,

$$\mathbb{R}_{*+}^{n \times n} = \mathbb{R}_+^{n \times n} + \mathbb{R}_d^{n \times n} \equiv \left\{ A + B \mid A \in \mathbb{R}_+^{n \times n}, B \in \mathbb{R}_d^{n \times n} \right\}.$$

Further, for $n = m = 1$, one has

$$\mathbb{R}_{*+}^{1 \times 1} = \mathbb{R}_d^{1 \times 1} = \mathbb{R}, \quad \mathbb{R}_+^{1 \times 1} = \mathbb{R}_+ \equiv [0, \infty). \quad (2.1)$$

We have the following simple result whose proof is obvious.

Proposition 2.1. *Let $A \in \mathbb{R}^{n \times m}$. Then $A \in \mathbb{R}_+^{n \times m}$ if and only if*

$$Ax \geq 0, \quad \forall x \geq 0. \quad (2.2)$$

Next, we introduce some spaces. Let $H = \mathbb{R}^n, \mathbb{R}^{n \times m}$, etc. with $|\cdot|$ beng its norm. For $1 \leq p, q < \infty$ and $0 \leq s < t \leq T$, define

$$\begin{aligned} L_{\mathcal{F}_t}^p(\Omega; H) &= \left\{ \xi : \Omega \rightarrow H \mid \xi \text{ is } \mathcal{F}_t\text{-measurable, } \mathbb{E}|\xi|^p < \infty \right\}, \\ L_{\mathcal{F}_T}^p(\Omega; L^q(s, t; H)) &= \left\{ X : [s, t] \times \Omega \rightarrow H \mid X(\cdot) \text{ is } \mathcal{F}_T\text{-measurable, } \mathbb{E} \left(\int_s^t |X(r)|^q dr \right)^{\frac{p}{q}} < \infty \right\}, \\ L_{\mathcal{F}_T}^p(\Omega; C([s, t]; H)) &= \left\{ X : [s, t] \times \Omega \rightarrow H \mid X(\cdot) \text{ is } \mathcal{F}_T\text{-measurable, has continuous paths,} \right. \\ &\quad \left. \mathbb{E} \left(\sup_{r \in [s, t]} |X(r)|^p \right) < \infty \right\}, \\ L_{\mathcal{F}_T}^q(s, t; L^p(\Omega; H)) &= \left\{ X : [s, t] \times \Omega \rightarrow H \mid X(\cdot) \text{ is } \mathcal{F}_T\text{-measurable, } \int_s^t \left(\mathbb{E} |X(r)|^p \right)^{\frac{q}{p}} dr < \infty \right\}, \\ C_{\mathcal{F}_T}([s, t]; L^p(\Omega; H)) &= \left\{ X : [s, t] \rightarrow L_{\mathcal{F}_T}^p(\Omega; H) \mid X(\cdot) \text{ is continuous, } \sup_{r \in [s, t]} \mathbb{E} |X(r)|^p < \infty \right\}. \end{aligned}$$

The spaces with the above p and/or q replaced by ∞ can be defined in an obvious way. Also, we define

$$L_{\mathbb{F}}^p(\Omega; L^q(s, t; H)) = \left\{ X(\cdot) \in L_{\mathcal{F}_T}^p(\Omega; L^q(s, t; H)) \mid X(\cdot) \text{ is } \mathbb{F}\text{-adapted} \right\}.$$

The spaces $L_{\mathbb{F}}^p(\Omega; C([s, t]; H))$, $L_{\mathbb{F}}^q(s, t; L^p(\Omega; H))$, and $C_{\mathbb{F}}([s, t]; L^p(\Omega; H))$ (with $1 \leq p, q \leq \infty$) can be defined in the same way. For simplicity, we denote

$$L_{\mathbb{F}}^p(s, t; H) = L_{\mathbb{F}}^p(\Omega; L^p(s, t; H)) = L_{\mathbb{F}}^p(s, t; L^p(\Omega; H)), \quad 1 \leq p \leq \infty.$$

Further, we denote

$$\Delta = \left\{ (t, s) \in [0, T]^2 \mid t \leq s \right\}, \quad \Delta^* = \left\{ (t, s) \in [0, T]^2 \mid t \geq s \right\} \equiv \overline{\Delta^c},$$

and let

$$\begin{aligned} L_{\mathbb{F}}^p(\Delta; H) &= \left\{ Z : \Delta \times \Omega \rightarrow H \mid s \mapsto Z(t, s) \text{ is } \mathbb{F}\text{-progressively measurable on } [t, T], \forall t \in [0, T], \right. \\ &\quad \left. \int_0^T \mathbb{E} \left(\int_t^T |Z(t, s)|^2 ds \right)^{\frac{p}{2}} dt < \infty \right\}, \end{aligned}$$

$$\begin{aligned} L^p(0, T; L_{\mathbb{F}}^2(0, T; H)) &= \left\{ Z : [0, T]^2 \times \Omega \rightarrow H \mid s \mapsto Z(t, s) \text{ is } \mathbb{F}\text{-progressively measurable} \right. \\ &\quad \left. \text{on } [t, T], \forall t \in [0, T], \int_0^T \mathbb{E} \left(\int_0^T |Z(t, s)|^2 ds \right)^{\frac{p}{2}} dt < \infty \right\}. \end{aligned}$$

The spaces $L_{\mathbb{F}}^{\infty}(\Delta; H)$ and $L^{\infty}(0, T; L_{\mathbb{F}}^2(0, T; H))$ can be defined similarly. Then we denote

$$\begin{cases} \mathcal{H}_{\Delta}^p[0, T] = L_{\mathbb{F}}^p(0, T) \times L_{\mathbb{F}}^p(\Delta; \mathbb{R}^n), \\ \mathcal{H}^p[0, T] = L_{\mathbb{F}}^p(0, T) \times L^p(0, T; L_{\mathbb{F}}^2(0, T; \mathbb{R}^n)), \\ \mathcal{M}^p[0, T] = \left\{ (y(\cdot), z(\cdot, \cdot)) \in \mathcal{H}^p[0, T] \mid y(t) = \mathbb{E}y(t) + \int_0^t z(t, s) dW(s), \ t \in [0, T] \right\}. \end{cases}$$

2.1 Comparison of solutions to FSDEs.

For any $(s, x) \in [0, T) \times \mathbb{R}^n$, let us first consider the following linear FSDE:

$$\begin{cases} dX(t) = (A_0(t)X(t) + b(t))dt + A_1(t)X(t)dW(t), \quad t \in [s, T], \\ X(s) = x, \end{cases} \quad (2.3)$$

with $A_0(\cdot)$ and $A_1(\cdot)$ satisfying the following assumption.

(FD1) The maps $A_0(\cdot), A_1(\cdot) \in L_{\mathbb{F}}^{\infty}(\Omega; C([0, T]; \mathbb{R}^{n \times n}))$.

We point out here that if the diffusion in (2.3) is replaced by $A_1(t)X(t) + \sigma(t)$ for some $\sigma(\cdot) \neq 0$, then comparison theorem might fail in general. Therefore, we restrict ourselves to the above form. It is standard that under (FD1), for any $(s, x) \in [0, T) \times \mathbb{R}^n$, $b(\cdot) \in L_{\mathbb{F}}^2(\Omega; L^1(s, T; \mathbb{R}^n))$, FSDE (2.3) admits a unique solution $X(\cdot) \equiv X(\cdot; s, x, b(\cdot)) \in L_{\mathbb{F}}^2(\Omega; C([s, T]; \mathbb{R}^n))$, and the following estimate holds:

$$\mathbb{E} \left[\sup_{t \in [s, T]} |X(t)|^2 \right] \leq K \left\{ |x|^2 + \mathbb{E} \left(\int_s^T |b(t)| dt \right)^2 \right\}. \quad (2.4)$$

Hereafter, $K > 0$ represents a generic constant which can be different from line to line. Let $\Phi(\cdot, \cdot)$ be the *stochastic fundamental matrix* of $\{A_0(\cdot), A_1(\cdot)\}$, i.e.,

$$\begin{cases} d\Phi(t, s) = A_0(t)\Phi(t, s)dt + A_1(t)\Phi(t, s)dW(t), \quad t \in [s, T], \\ \Phi(s, s) = I. \end{cases} \quad (2.5)$$

Then one has the following *variation of constant formula*:

$$X(t; s, x) = \Phi(t, s)x + \int_s^t \Phi(t, \tau)b(\tau)d\tau, \quad t \in [s, T], \quad (2.6)$$

for the solution $X(\cdot) \equiv X(\cdot; s, x, b(\cdot))$ of (2.3). We have the following result.

Proposition 2.2. *Let (FD1) hold. Then the stochastic fundamental matrix $\Phi(\cdot, \cdot)$ of $\{A_0(\cdot), A_1(\cdot)\}$ satisfies the following:*

$$\Phi(t, s)x \geq 0, \quad \forall x \geq 0, \quad 0 \leq s \leq t \leq T, \text{ a.s. }, \quad (2.7)$$

if and only if

$$A_0(t) \in \mathbb{R}_{*+}^{n \times n}, \quad t \in [0, T], \text{ a.s. }, \quad (2.8)$$

and

$$A_1(t) \in \mathbb{R}_d^{n \times n}, \quad t \in [0, T], \text{ a.s. } \quad (2.9)$$

Consequently, in this case, for any $(s, x) \in [0, T) \times \mathbb{R}^n$ and $b(\cdot) \in L_{\mathbb{F}}^2(\Omega; L^1(s, T; \mathbb{R}^n))$ with

$$x \geq 0, \quad b(t) \geq 0, \quad \text{a.e. } t \in [s, T], \text{ a.s. }, \quad (2.10)$$

the unique solution $X(\cdot) \equiv X(\cdot; s, x, b(\cdot)) \in L_{\mathbb{F}}^2(\Omega; C([s, T]; \mathbb{R}^n))$ of linear FSDE (2.3) corresponding to $(x, b(\cdot))$ on $[s, T]$ satisfies the following:

$$X(t) \geq 0, \quad \forall t \in [s, T], \text{ a.s. } \quad (2.11)$$

The above result should be known (at least for the case $n = 1$). For reader's convenience, we provide a proof here, which is straightforward.

Proof. Sufficiency. Let $X(\cdot) \equiv X(\cdot; s, x, 0)$ be the solution to linear FSDE (2.3) with $(s, x) \in [0, T) \times \mathbb{R}^n$ and $b(\cdot) = 0$. Then

$$X(t) = \Phi(t, s)x, \quad 0 \leq s \leq t \leq T.$$

It suffices to show that $x \leq 0$ implies

$$X(t) \leq 0, \quad t \in [s, T], \quad \text{a.s.} \quad (2.12)$$

To prove (2.12), we define a convex function

$$f(x) = \sum_{i=1}^n (x_i^+)^2, \quad \forall x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n,$$

where $a^+ = \max\{a, 0\}$ for any $a \in \mathbb{R}$. Applying Itô's formula to $f(X(t))$, we get

$$\begin{aligned} f(X(t)) - f(x) &= \int_s^t \left[\langle f_x(X(\tau)), A_0(\tau)X(\tau) \rangle + \frac{1}{2} \langle f_{xx}(X(\tau))A_1(\tau)X(\tau), A_1(\tau)X(\tau) \rangle \right] d\tau \\ &\quad + \int_s^t \langle f_x(X(\tau)), A_1(\tau)X(\tau) \rangle dW(\tau). \end{aligned}$$

Let us observe the following: (noting $A_0(\tau) \in \mathbb{R}_{*+}^{n \times n}$)

$$\begin{aligned} \langle f_x(X(\tau)), A_0(\tau)X(\tau) \rangle &= \sum_{i,j=1}^n 2X_i(\tau)^+ \langle e_i, A_0(\tau)e_j \rangle X_j(\tau) \\ &= \sum_{i=1}^n 2X_i(\tau)^+ \langle e_i, A_0(\tau)e_i \rangle X_i(\tau) + \sum_{i \neq j} 2X_i(\tau)^+ \langle e_i, A_0(\tau)e_j \rangle X_j(\tau) \\ &\leq \sum_{i=1}^n 2[X_i(\tau)^+]^2 \langle e_i, A_0(\tau)e_i \rangle + \sum_{i \neq j} 2 \langle e_i, A_0(\tau)e_j \rangle X_i(\tau)^+ X_j(\tau)^+ \leq Kf(X(\tau)). \end{aligned}$$

Next, we have (noting $A_1(\cdot)$ and $f_{xx}(\cdot)$ are diagonal)

$$\begin{aligned} \frac{1}{2} \mathbb{E} \langle f_{xx}(X(\tau))A_1(\tau)X(\tau), A_1(\tau)X(\tau) \rangle &= \frac{1}{2} \mathbb{E} \sum_{i=1}^n I_{(X_i(\tau) \geq 0)} \left(\langle A_1(\tau)e_i, e_i \rangle X_i(\tau) \right)^2 \\ &= \frac{1}{2} \mathbb{E} \sum_{i=1}^n \langle A_1(\tau)e_i, e_i \rangle^2 [X_i(\tau)^+]^2 \leq Kf(X(\tau)). \end{aligned}$$

Consequently,

$$\mathbb{E}f(X(t)) \leq f(x) + K \int_s^t \mathbb{E}f(X(\tau))d\tau, \quad t \in [s, T].$$

Hence, by Gronwall's inequality, we obtain

$$\sum_{i=1}^n \mathbb{E}|X_i(t)^+|^2 \leq K \sum_{i=1}^n |x_i^+|^2, \quad t \in [s, T].$$

Therefore, if $x \leq 0$, then

$$\sum_{i=1}^n \mathbb{E}|X_i(t)^+|^2 = 0, \quad \forall t \in [s, T].$$

This leads to (2.12).

Necessity. Let

$$\Omega_{ij}^0(s) = \{\omega \in \Omega \mid \langle A_0(s)e_i, e_j \rangle < 0\}, \quad 1 \leq i, j \leq n, \quad s \in [0, T].$$

Suppose (2.8) fails. Then for some $i \neq j$, and some $s \in [0, T]$,

$$\mathbb{P}(\Omega_{ij}^0(s)) > 0,$$

i.e., the (j, i) -th (off-diagonal) entry of $A_0(s)$ is not almost surely nonnegative. Let $X(\cdot) \equiv X(\cdot; s, e_i, 0)$ be the solution to linear FSDE (2.3) with $(s, x, b(\cdot)) = (s, e_i, 0)$. Then

$$\begin{aligned} \mathbb{E}[X_j(t)I_{\Omega_{ij}^0(s)}] &= \mathbb{E}[\langle X(t), e_j \rangle I_{\Omega_{ij}^0(s)}] = \int_s^t \mathbb{E}[I_{\Omega_{ij}^0(s)} \langle A_0(\tau)X(\tau), e_j \rangle] d\tau \\ &= \mathbb{E}[\langle A_0(s)e_i, e_j \rangle I_{\Omega_{ij}^0(s)}](t-s) + o(t-s) < 0, \end{aligned}$$

for $t-s > 0$ small. Thus, $X(t) = \Phi(t, s)e_i \geq 0$ fails for some $t \in [s, T]$ that is close to s . This shows that (2.8) is necessary.

Next, suppose (2.9) fails, i.e.,

$$\mathbb{P}(\langle A_1(s)e_i, e_j \rangle \neq 0) > 0,$$

for some $i \neq j$, and $s \in [0, T]$, i.e., the (j, i) -th (off-diagonal) entry of $A_1(s)$ is not identically equal to zero. Let $\Phi_0(\cdot, \cdot)$ be the fundamental matrix of $A_0(\cdot)$, i.e.,

$$\Phi_0(t, s) = I + \int_s^t A_0(\tau)\Phi_0(\tau, s)d\tau, \quad 0 \leq s \leq t \leq T.$$

Then $\Phi_0(\cdot, \cdot)^{-1}$ satisfies

$$\Phi_0(t, s)^{-1} = I - \int_s^t \Phi_0(\tau, s)^{-1}A_0(\tau)d\tau, \quad 0 \leq s \leq t \leq T.$$

Hence,

$$|\Phi_0(t, s)^{-1} - I| \leq K(t-s), \quad 0 \leq s \leq t \leq T, \quad \text{a.s.}$$

Now, let $X(\cdot) = X(\cdot; s, e_i, 0)$. Then

$$X(t) = \Phi_0(t, s) \left[e_i + \int_s^t \Phi_0(\tau, s)^{-1}A_1(\tau)X(\tau)dW(\tau) \right], \quad 0 \leq s \leq t \leq T.$$

Thus, for $j \neq i$,

$$\langle \Phi_0(t, s)^{-1}X(t), e_j \rangle = \int_s^t \langle \Phi_0(\tau, s)^{-1}A_1(\tau)X(\tau), e_j \rangle dW(\tau), \quad t \in [s, T].$$

Consequently,

$$\begin{aligned} X_j(t) &= \langle X(t), e_j \rangle = \langle [I - \Phi_0(t, s)^{-1}]X(t), e_j \rangle + \int_s^t \langle \Phi_0(\tau, s)^{-1}A_1(\tau)X(\tau), e_j \rangle dW(\tau) \\ &= \langle [I - \Phi_0(t, s)^{-1}]X(t), e_j \rangle + \langle A_1(s)e_i, e_j \rangle [W(t) - W(s)] \\ &\quad + \int_s^t \langle A_1(s)e_i - \Phi_0(\tau, s)^{-1}A_1(\tau)X(\tau), e_j \rangle dW(\tau). \end{aligned}$$

Note that

$$\mathbb{E}|\langle [I - \Phi_0(t, s)^{-1}]X(t), e_j \rangle| \leq \mathbb{E}[|I - \Phi_0(t, s)| |X(t)|] \leq K(t-s).$$

Also,

$$\begin{aligned} & \mathbb{E} \left| \int_s^t \langle A_1(s)e_i - \Phi_0(\tau, s)^{-1}A_1(\tau)X(\tau), e_j \rangle dW(\tau) \right| \\ & \leq K \mathbb{E} \left(\int_s^t |A_1(s)e_i - \Phi_0(\tau, s)^{-1}A_1(\tau)X(\tau)|^2 d\tau \right)^{\frac{1}{2}} = o((t-s)^{\frac{1}{2}}). \end{aligned}$$

Therefore,

$$\begin{cases} |\mathbb{E}X_j(t)|^2 = o(t-s), \\ \mathbb{E}|X_j(t)|^2 = \mathbb{E}|\langle A_1(s)e_i, e_j \rangle|^2(t-s) - o(t-s). \end{cases} \quad (2.13)$$

If we let

$$X_j(t)^+ = X_j(t) \vee 0, \quad X_j(t)^- = [-X_j(t)] \vee 0,$$

then

$$X_j(t) = X_j(t)^+ - X_j(t)^-, \quad |X_j(t)| = X_j(t)^+ + X_j(t)^-.$$

Consequently, (2.13) can be written as

$$\begin{cases} (\mathbb{E}X_j(t)^+ - \mathbb{E}X_j(t)^-)^2 = o(t-s), \\ \mathbb{E}[X_j(t)^+]^2 + \mathbb{E}[X_j(t)^-]^2 = \mathbb{E}|\langle A_1(s)e_i, e_j \rangle|^2(t-s) - o(t-s). \end{cases}$$

Hence, it is necessary that

$$\mathbb{E}[X_j(t)^+]^2, \quad \mathbb{E}[X_j(t)^-]^2 > 0,$$

as long as $t-s > 0$ is small, which implies

$$\mathbb{P}(X_j(t) < 0) > 0,$$

a contradiction. \square

We point out that in the above, the dimension $n \geq 1$; and if $n = 1$, conditions (2.8)–(2.9) are automatically true.

Now, let us look at the following general nonlinear FSDEs, in their integral form: For $i = 0, 1$,

$$X^i(t) = x^i + \int_s^t b^i(r, X^i(r))dr + \int_s^t \sigma(r, X^i(r))dW(r), \quad t \in [s, T]. \quad (2.14)$$

Note that unlike the drift $b^i(r, x)$, the diffusion $\sigma(r, x)$ is independent of $i = 0, 1$. We introduce the following assumption.

(FD2) For $i = 0, 1$, the maps $b^i, \sigma : [0, T] \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ are measurable, $t \mapsto (b^i(t, x), \sigma(t, x))$ is \mathbb{F} -progressively measurable, $x \mapsto (b^i(t, x), \sigma(t, x))$ is uniformly Lipschitz, and $t \mapsto (b^i(t, 0), \sigma(t, 0))$ is uniformly bounded.

It is standard that under (FD2), for any $(s, x^i) \in [0, T] \times \mathbb{R}^n$, (2.14) admits a unique strong solution $X^i(\cdot) \equiv X^i(\cdot; s, x^i)$. We have the following comparison theorem.

Theorem 2.3. *Let (FD2) hold. Suppose $\bar{b} : [0, T] \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ is measurable, $t \mapsto \bar{b}(t, x)$ is \mathbb{F} -progressively measurable, $\bar{b}_x(t, x)$ exists and is uniformly bounded.*

(i) *Let*

$$\begin{cases} \bar{b}_x(t, x) \in \mathbb{R}_{*+}^n, \\ \sigma_x(t, x) \in \mathbb{R}_d^{n \times n}, \end{cases} \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n, \text{ a.s.} \quad (2.15)$$

Suppose

$$b^0(t, x) \leq \bar{b}(t, x) \leq b^1(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n, \text{ a.s.}, \quad (2.16)$$

Then for any $(s, x^i) \in [0, T] \times \mathbb{R}^n$ with

$$x^0 \leq x^1,$$

the unique solutions $X^i(\cdot) \equiv X^i(\cdot; s, x^i)$ of (2.14) satisfy

$$X^0(t) \leq X^1(t), \quad t \in [s, T], \text{ a.s.} \quad (2.17)$$

(ii) Suppose

$$b^0(t, x) = \bar{b}(t, x) = b^1(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n, \text{ a.s.},$$

and $(t, x) \mapsto (\bar{b}(t, x), \sigma(t, x))$ is continuous. Then (2.15) is necessary for the conclusion of (i) to hold.

Proof. (i) Let $\bar{x} \in \mathbb{R}^n$ with

$$x^0 \leq \bar{x} \leq x^1.$$

Let $\bar{X}(\cdot)$ be the solution to the following FSDE:

$$\bar{X}(t) = \bar{x} + \int_s^t \bar{b}(r, \bar{X}(r))dr + \int_s^t \sigma(r, \bar{X}(r))dW(r), \quad t \in [s, T].$$

Then

$$\begin{aligned} \bar{X}(t) - X^0(t) &= \bar{x} - x^0 + \int_s^t [\bar{b}(r, X^0(r)) - b^0(r, X^0(r))]dr \\ &\quad + \int_s^t \bar{b}_x(r) [\bar{X}(r) - X^0(r)]dr + \int_s^t \sigma_x(r) [\bar{X}(r) - X^0(r)]dW(r), \end{aligned}$$

where

$$\begin{aligned} \bar{b}_x(r) &= \int_0^1 \bar{b}_x(r, X^0(r) + \lambda[\bar{X}(r) - X^0(r)])d\lambda \in \mathbb{R}_{*+}^{n \times n}, \\ \sigma_x(r) &= \int_0^1 \sigma_x(r, X^0(r) + \lambda[\bar{X}(r) - X^0(r)])d\lambda \in \mathbb{R}_d^{n \times n}. \end{aligned}$$

Hence, by Proposition 2.2, we obtain

$$X^0(t) \leq \bar{X}(t), \quad t \in [0, T], \text{ a.s.}$$

Similarly, we are able to show that

$$\bar{X}(t) \leq X^1(t), \quad t \in [0, T], \text{ a.s.}$$

Then (2.17) follows.

(ii) For any $x, \tilde{x} \in \mathbb{R}^n$ and $\tilde{x} \geq 0, \delta \geq 0$, let $X^\delta(\cdot)$ be the solution to the following:

$$X^\delta(t) = x + \delta\tilde{x} + \int_s^t \bar{b}(r, X^\delta(r))dr + \int_s^t \sigma(r, X^\delta(r))dW(r), \quad t \in [s, T],$$

and $\tilde{X}(\cdot)$ be the solution to the following:

$$\tilde{X}(t) = \tilde{x} + \int_s^t \bar{b}_x(r, X^0(r))\tilde{X}(r)dr + \int_s^t \sigma_x(r, X^0(r))\tilde{X}(r)dW(r), \quad t \in [s, T].$$

Then it is straightforward that

$$\tilde{X}(t) = \lim_{\delta \rightarrow 0} \frac{X^\delta(t) - X^0(t)}{\delta}, \quad t \in [s, T], \text{ a.s.}$$

Hence, the conclusion of (i) implies that

$$\tilde{X}(t) \geq 0, \quad \forall t \in [s, T], \text{ a.s.}$$

Then by Proposition 2.2, we must have

$$\bar{b}_x(r, X^0(r)) \in \mathbb{R}_{*+}^{n \times n}, \quad \sigma_x(r, X^0(r)) \in \mathbb{R}_d^{n \times n}, \quad r \in [s, T], \text{ a.s.}$$

Setting $r = s$, we obtain (2.15). □

2.2 Comparison of adapted solutions to BSDEs.

We now look at the following n -dimensional linear BSDE:

$$\begin{cases} dY(t) = [A(t)Y(t) + B(t)Z(t) - g(t)]dt + Z(t)dW(t), & t \in [0, \tau], \\ Y(\tau) = \xi, \end{cases} \quad (2.18)$$

where $\xi \in L^2_{\mathcal{F}_\tau}(\Omega; \mathbb{R}^n)$, with τ being an \mathbb{F} -stopping time taking values in $(0, T]$. The same as (FD1), we introduce the following hypothesis.

(BD1) The maps $A(\cdot), B(\cdot) \in L^\infty_{\mathbb{F}}(\Omega; C([0, T]; \mathbb{R}^{n \times n}))$.

The following is comparable with Proposition 2.2.

Proposition 2.4. *Let (BD1) hold. Then for any \mathbb{F} -stopping time τ valued in $(0, T]$, any $g(\cdot) \in L^2_{\mathbb{F}}(0, \tau; \mathbb{R}^n)$ and $\xi \in L^2_{\mathcal{F}_\tau}(\Omega; \mathbb{R}^n)$ with*

$$\xi \geq 0, \quad g(t) \geq 0, \quad \text{a.e. } t \in [0, \tau], \text{ a.s.}, \quad (2.19)$$

the adapted solution $(Y(\cdot), Z(\cdot))$ to BSDE (2.18) satisfies

$$Y(t) \geq 0, \quad t \in [0, \tau], \text{ a.s.},$$

if and only if

$$-A(t) \in \mathbb{R}^{n \times n}_{*+}, \quad B(t) \in \mathbb{R}^{n \times n}_d, \quad t \in [0, T], \text{ a.s.} \quad (2.20)$$

Proof. Sufficiency. Let s, τ be any \mathbb{F} -stopping times such that $0 \leq s < \tau \leq T$, almost surely. For any $x \in \mathbb{R}^n$, let $X(\cdot)$ be the strong solution to the following FSDE:

$$\begin{cases} dX(t) = -A(t)^T X(t)dt - B(t)^T X(t)dW(t), & t \in [s, \tau], \\ X(s) = x. \end{cases} \quad (2.21)$$

We claim that the following duality relation holds:

$$\langle x, Y(s) \rangle = \mathbb{E}_s \left[\langle X(\tau), \xi \rangle + \int_s^\tau \langle X(r), g(r) \rangle dr \right], \quad (2.22)$$

where $\mathbb{E}_s[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_s]$. In fact, by Itô's formula,

$$\begin{aligned} \mathbb{E}_s \left[\langle X(\tau), \xi \rangle - \langle x, Y(s) \rangle \right] &= \mathbb{E}_s \int_s^\tau \left[-\langle A(r)^T X(r), Y(r) \rangle + \langle X(r), A(r)Y(r) + B(r)Z(r) - g(r) \rangle \right. \\ &\quad \left. - \langle B(r)^T X(r), Z(r) \rangle \right] dr = -\mathbb{E}_s \int_s^\tau \langle X(r), g(r) \rangle dr. \end{aligned}$$

Hence, (2.22) follows.

Now, for any $x \in \mathbb{R}^n_+$, under our conditions, by Proposition 2.2, the solution $X(\cdot)$ of (2.21) satisfies

$$X(t) \geq 0, \quad t \in [s, \tau], \text{ a.s.}$$

Hence, by duality relation (2.22),

$$\langle x, Y(s) \rangle = \mathbb{E}_s \left[\langle X(\tau), \xi \rangle + \int_s^\tau \langle X(r), g(r) \rangle dr \right] \geq 0,$$

proving our conclusion.

Necessity. Suppose (2.20) fails. Then, by Proposition 2.2, for some $i \neq j$ and $s \in [0, T]$, the solution $X(\cdot) \equiv X(\cdot; s, e_i)$ of (2.21) satisfies

$$\mathbb{P}(X_j(\tau) < 0) > 0,$$

for some $\tau > s$. For such a τ , choosing $\xi = e_j I_{\{X_j(\tau) < 0\}}$, and $g(\cdot) = 0$, we have

$$Y_i(s) = \langle e_i, Y(s) \rangle = \mathbb{E}_s \left[\langle X(\tau), e_j \rangle I_{\{X_j(\tau) < 0\}} \right] = \mathbb{E}_s \left[X_j(\tau) I_{\{X_j(\tau) < 0\}} \right] < 0,$$

a contradiction. \square

We now look at nonlinear n -dimensional BSDEs: For $i = 0, 1$, and \mathbb{F} -stopping time τ valued in $[0, T]$,

$$Y^i(t) = \xi^i + \int_t^T g^i(s, Y^i(s), Z^i(s)) ds - \int_t^T Z^i(s) dW(s), \quad t \in [0, \tau]. \quad (2.23)$$

Let us introduce the following standard assumption.

(BD2) For $i = 0, 1$, the map $g^i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ is measurable, $s \mapsto g^i(s, y, z)$ is \mathbb{F} -progressively measurable, $(y, z) \mapsto g^i(s, y, z)$ is uniformly Lipschitz, $s \mapsto g^i(s, 0, 0)$ is uniformly bounded.

It is well-known that under (BD2), for any $\xi^i \in L^p_{\mathcal{F}_\tau}(\Omega; \mathbb{R}^n)$ (with $p > 1$), BSDE (2.23) admits a unique adapted solution $(Y^i(\cdot), Z^i(\cdot))$. Based on Proposition 2.4, we have the following comparison theorem for nonlinear n -dimensional BSDEs.

Theorem 2.5. *Let (BD2) hold. Suppose $\bar{g} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ is measurable, $s \mapsto \bar{g}(s, y, z)$ is \mathbb{F} -progressively measurable, $\bar{g}_y(s, y, z)$ and $\bar{g}_z(s, y, z)$ exist and are uniformly bounded.*

(i) *Suppose*

$$\bar{g}_y(s, y, z) \in \mathbb{R}^{n \times n}_{*+}, \quad \bar{g}_z(s, y, z) \in \mathbb{R}^{n \times n}_d, \quad \forall (s, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n, \text{ a.s.}, \quad (2.24)$$

and

$$g^0(s, y, z) \leq \bar{g}(s, y, z) \leq g^1(s, y, z), \quad \forall (s, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n, \text{ a.s.} \quad (2.25)$$

Then for any \mathbb{F} -stopping time τ valued in $(0, T]$, and any $\xi^0, \xi^1 \in L^2_{\mathcal{F}_\tau}(\Omega; \mathbb{R}^n)$ with

$$\xi^0 \leq \xi^1, \quad \text{a.s.},$$

the corresponding adapted solutions $(Y^i(\cdot), Z^i(\cdot))$ of BSDEs (2.23) satisfy

$$Y^0(t) \leq Y^1(t), \quad t \in [0, \tau], \text{ a.s.} \quad (2.26)$$

(ii) *Suppose*

$$g^0(s, y, z) = \bar{g}(s, y, z) = g^1(s, y, z), \quad \forall (s, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n, \text{ a.s.},$$

and $(s, y, z) \mapsto \bar{g}(s, y, z)$ is continuous. Then (2.24) is necessary for the conclusion of (i) to be true.

Proof. (i) Let $\bar{\xi} \in L^2_{\mathcal{F}_\tau}(\Omega; \mathbb{R}^n)$ such that

$$\xi^0 \leq \bar{\xi} \leq \xi^1, \quad \text{a.s.}$$

Let $(\bar{Y}(\cdot), \bar{Z}(\cdot))$ be the adapted solution to the following BSDE:

$$\bar{Y}(t) = \bar{\xi} + \int_t^\tau \bar{g}(s, \bar{Y}(s), \bar{Z}(s)) ds - \int_t^\tau \bar{Z}(s) dW(s), \quad t \in [0, \tau].$$

Observe

$$\begin{aligned}\bar{Y}(t) - Y^0(t) &= \bar{\xi} - \xi^0 + \int_t^\tau [\bar{g}(s, Y^0(s), Z^0(s)) - g^0(s, Y^0(s), Z^0(s))] ds \\ &\quad + \int_t^\tau [A(s)(\bar{Y}(s) - Y^0(s)) + B(s)(\bar{Z}(s) - Z^0(s))] ds - \int_t^\tau (\bar{Z}(s) - Z^0(s)) dW(s),\end{aligned}$$

where

$$\begin{aligned}A(s) &= \int_0^1 \bar{g}_y(s, Y^0(s) + \beta[\bar{Y}(s) - Y^0(s)], \beta[\bar{Z}(s) - Z^0(s)]) d\beta \in \mathbb{R}_{*+}^{n \times n}, \\ B(s) &= \int_0^1 \bar{g}_z(s, Y^0(s) + \beta[\bar{Y}(s) - Y^0(s)], \beta[\bar{Z}(s) - Z^0(s)]) d\beta \in \mathbb{R}_d^{n \times n}.\end{aligned}$$

Hence, by Proposition 2.4, we obtain our conclusion.

(ii) For any given deterministic $\tau \in [0, T]$, any $\xi, \tilde{\xi} \in L_{\mathcal{F}_\tau}^2(\Omega; \mathbb{R}^n)$ and $\tilde{\xi} \geq 0, \delta \geq 0$, let $(Y^\delta(\cdot), Z^\delta(\cdot))$ be the adapted solution to the following BSDE:

$$Y^\delta(t) = \xi + \delta \tilde{\xi} + \int_t^\tau \bar{g}(s, Y^\delta(s), Z^\delta(s)) ds - \int_t^\tau Z^\delta(s) dW(s), \quad t \in [0, \tau].$$

In particular,

$$Y^0(t) = \xi + \int_t^\tau \bar{g}(s, Y^0(s), Z^0(s)) ds - \int_t^\tau Z^0(s) dW(s), \quad t \in [0, \tau]. \quad (2.27)$$

If we let $(\tilde{Y}(\cdot), \tilde{Z}(\cdot))$ be the adapted solution to the following BSDE:

$$\tilde{Y}(t) = \tilde{\xi} + \int_t^\tau (\bar{g}_y(s, Y^0(s), Z^0(s)) \tilde{Y}(s) + \bar{g}_z(s, Y^0(s), Z^0(s)) \tilde{Z}(s)) ds - \int_t^\tau \tilde{Z}(s) dW(s), \quad t \in [0, \tau],$$

then it is ready to show that

$$\lim_{\delta \rightarrow 0} \frac{Y^\delta(t) - Y^0(t)}{\delta} = \tilde{Y}(t), \quad \lim_{\delta \rightarrow 0} \frac{Z^\delta(t) - Z^0(t)}{\delta} = \tilde{Z}(t), \quad t \in [0, \tau], \text{ a.s.}$$

Hence, conclusion of (i) implies that

$$\tilde{Y}(t) \geq 0, \quad \forall t \in [0, \tau], \text{ a.s.}$$

Consequently, by Proposition 2.4, we obtain

$$\bar{g}_y(s, Y^0(s), Z^0(s)) \in \mathbb{R}_{*+}^{n \times n}, \quad \bar{g}_z(s, Y^0(s), Z^0(s)) \in \mathbb{R}_d^{n \times n}, \quad s \in [0, \tau], \text{ a.s.}, \quad (2.28)$$

for the adapted solution $(Y^0(\cdot), Z^0(\cdot))$ of BSDE (2.27) with any $\xi \in L_{\mathcal{F}_\tau}^2(\Omega; \mathbb{R}^n)$. Now let $\tau = T$. For any $s \in [0, T)$ and $y, z \in \mathbb{R}^n$, let

$$\xi = y + z[W(T) - W(s)] - \int_s^T \bar{g}(r, \bar{Y}^0(r), z) dr,$$

where $\bar{Y}^0(\cdot)$ is the unique solution of (forward) Volterra integral equation

$$\bar{Y}^0(t) = y + z[W(t) - W(s)] - \int_s^t \bar{g}(r, \bar{Y}^0(r), z) dr, \quad t \in [s, T].$$

Then it is easy to show that $(\bar{Y}^0(\cdot), z)$ is the unique adapted solution to the following BSDE

$$Y^0(t) = \xi + \int_t^T \bar{g}(r, Y^0(r), Z^0(r)) dr - \int_t^T Z^0(r) dW(r), \quad t \in [s, T].$$

Clearly, $Y^0(s) = y$, and from (2.28), we have

$$\bar{g}_y(s, y, z) \in \mathbb{R}_{*+}^{n \times n}, \quad \bar{g}_z(s, y, z) \in \mathbb{R}_d^{n \times n}, \quad \text{a.s.}$$

Hence, (2.24) follows. \square

The above result is a slight extension of a relevant one presented in [14], allowing $g^0(\cdot)$ and $g^1(\cdot)$ to be different for the sufficient part. Note that as long as the map $\bar{g}(\cdot)$ exists satisfying (2.24) and (3.9), we allow the j -th component of $g^i(s, y, z)$ to depend on k -th component of Z with $k \neq j$. For example, suppose $\bar{g}(\cdot)$ satisfies (2.24). Then the comparison theorem holds for the case, say,

$$g^0(s, y, z) = \bar{g}(s, y, z) - |z|, \quad g^1(s, y, z) = \bar{g}(s, y, z) + |z|, \quad (s, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n.$$

Finally, we point out that our proof is based on the duality and a corresponding result for linear FSDs (Proposition 2.2), which is different from that found in [14].

2.3 Comparison of solutions to FSVIEs.

Let us now turn to FSVIEs. We consider the following linear FSVIE:

$$X(t) = \varphi(t) + \int_0^t (A_0(t, s)X(s) + b(s))ds + \int_0^t (A_1(t, s)X(s) + \sigma(s))dW(s), \quad t \in [0, T]. \quad (2.29)$$

Replacing $\varphi(\cdot)$ by

$$\varphi(\cdot) + \int_0^\cdot b(s)ds + \int_0^\cdot \sigma(s)dW(s),$$

we see that without loss of generality, it suffices to consider the following FSVIE:

$$X(t) = \varphi(t) + \int_0^t A_0(t, s)X(s)ds + \int_0^t A_1(t, s)X(s)dW(s), \quad t \in [0, T], \quad (2.30)$$

namely, we may assume $b(\cdot) = \sigma(\cdot) = 0$ in (2.29). We now look at a couple of examples which will help us to exclude some cases for which the comparison theorem may fail in general.

Example 2.6. Consider the following one-dimensional equation:

$$X(t) = 1 - 2e^t \int_0^t e^{-s}X(s)ds, \quad t \in [0, T].$$

In this case, we have

$$\varphi(t) = 1, \quad A_0(t, s) = -2e^{t-s}, \quad A_1(t, s) = 0, \quad \forall (t, s) \in \Delta^*.$$

To solve it, let

$$x(t) = \int_0^t e^{-s}X(s)ds, \quad t \in [0, T].$$

Then

$$\dot{x}(t) = e^{-t}X(t) = e^{-t} - 2x(t), \quad x(0) = 0.$$

Hence,

$$x(t) = \int_0^t e^{-2(t-s)}e^{-s}ds = e^{-2t}(e^t - 1) = e^{-t} - e^{-2t}, \quad t \in [0, T].$$

Therefore, the solution $X(\cdot)$ is given by

$$X(t) = 1 - 2e^t x(t) = 1 - 2e^t(e^{-t} - e^{-2t}) = -1 + 2e^{-t}, \quad t \in [0, T].$$

Consequently, for $T > \ln 2$, we have

$$X(T) = -1 + 2e^{-T} < 0.$$

This example shows that even for the deterministic case, i.e., $A_1(\cdot, \cdot) = 0$, the comparison of the solutions may fail. This is mainly due to the fact that $A_0(\cdot, \cdot)$ is negative and $t \mapsto A_0(t, s) \equiv -2e^{t-s}$ is decreasing.

Example 2.7. Consider the following one-dimensional FSVIE:

$$X(t) = 2T - t + \int_0^t X(s) dW(s), \quad t \in [0, T]. \quad (2.31)$$

Clearly, (2.31) is a special case of (2.29) with

$$\varphi(t) = 2T - t > 0, \quad A_0(t, s) = 0, \quad A_1(t, s) = 1.$$

Thus, $\varphi(\cdot)$ is (strictly) positive, and both $A_0(\cdot, \cdot)$ and $A_1(\cdot, \cdot)$ are constants. Note that (2.31) is equivalent to the following FSDE:

$$\begin{cases} dX(t) = -dt + X(t)dW(t), & t \in [0, T], \\ X(0) = 2T. \end{cases}$$

Therefore, the solution $X(\cdot)$ of the above satisfies the following:

$$X(t) = e^{-\frac{1}{2}t + W(t)} \left[2T - \int_0^t e^{\frac{1}{2}s - W(s)} ds \right] \leq e^{-\frac{1}{2}t + W(t)} \left[2T - \int_0^t e^{-W(s)} ds \right], \quad t \in [0, T]. \quad (2.32)$$

By the convexity of $\lambda \mapsto e^\lambda$, we have

$$\frac{1}{t} \int_0^t e^{-W(s)} ds \geq e^{-\frac{1}{t} \int_0^t W(s) ds}.$$

Thus, for any $t > 0$, $X(t) < 0$ is implied by

$$e^{-\frac{1}{t} \int_0^t W(s) ds} \geq \frac{2T}{t},$$

which is equivalent to the following:

$$-\frac{1}{t} \int_0^t W(s) ds \geq \log \frac{2T}{t}.$$

Since the left hand side of the above is a normal random variable, we therefore obtain

$$\mathbb{P}(X(t) < 0) \geq \mathbb{P}\left(-\frac{1}{t} \int_0^t W(s) ds \geq \log \frac{K}{t}\right) > 0. \quad (2.33)$$

This means that the comparison theorem fails for this example.

From the above, we see that when the diffusion is not identically zero, nonnegativity of the free term $\varphi(\cdot)$ is not enough to ensure the nonnegativity of the solution $X(\cdot)$ to FSVIE (2.30). The main reason for the comparison fails in this example is due to the fact that $t \mapsto \varphi(t)$ is decreasing. Next example is relevant to a result from [20], and it is simpler.

Example 2.8. Consider

$$X(t) = 1 + \int_0^t \frac{2T-s}{2T-t} X(s) dW(s), \quad t \in [0, T], \quad (2.34)$$

We see that the above is a special case of (2.30) with

$$\varphi(t) = 1, \quad A_0(t, s) = 0, \quad A_1(t, s) = \frac{2T-s}{2T-t}.$$

The main feature of the above is that the diffusion coefficient $A_1(t, s)$ depends on (t, s) and the variables t and s cannot be separated, meaning that $A_1(t, s)$ cannot be written as the product $A_{11}(t)A_{12}(s)$ of some single variable functions $A_{11}(\cdot)$ and $A_{12}(\cdot)$. Clearly, the process $\tilde{X}(t) \equiv (2T - t)X(t)$ satisfies the following FSVIE:

$$\tilde{X}(t) = 2T - t + \int_0^t \tilde{X}(s) dW(s), \quad t \in [0, T],$$

which coincides with (2.31). Hence, by Example 2.5, although the free term $\varphi(t) = 1 > 0$ in (2.34), we have

$$\mathbb{P}(X(t) < 0) > 0,$$

comparison theorem fails for (2.34).

The above example tells us that if $A_1(t, s)$ is not independent of t , even if the free term $\varphi(\cdot)$ is a constant, comparison theorem could fail in general. Therefore, if a linear FSVIE is considered for a general comparison theorem, we had better restrict ourselves to the following type:

$$X(t) = \varphi(t) + \int_0^t A_0(t, s)X(s)ds + \int_0^t A_1(s)X(s)dW(s), \quad t \in [0, T]. \quad (2.35)$$

To present positive results, we introduce the following assumption.

(FV1) The maps $A_0 : \Delta^* \times \Omega \rightarrow \mathbb{R}^{n \times n}$ and $A_1 : [0, T] \times \Omega \rightarrow \mathbb{R}^{n \times n}$ are measurable and uniformly bounded. For any $t \in [0, T]$, $s \mapsto (A_0(t, s), A_1(s))$ is \mathbb{F} -progressively measurable on $[0, t]$, and for any $s \in [0, T]$, the map $t \mapsto A_0(t, s)$ is continuous on $[s, T]$.

We present the following result.

Proposition 2.9. *Let (FV1) hold.*

(i) *Suppose*

$$A_0(t, s) \in \mathbb{R}_+^{n \times n}, \quad A_1(s) = 0, \quad \text{a.e. } (t, s) \in \Delta^*, \text{ a.s.} \quad (2.36)$$

Then for any $\varphi(\cdot) \in L_{\mathbb{F}}^2(0, T; \mathbb{R}^n)$ (2.35) admits a unique solution $X(\cdot) \in L_{\mathbb{F}}^2(0, T; \mathbb{R}^n)$ and it satisfies

$$X(t) \geq \varphi(t) \geq 0, \quad t \in [0, T]. \quad (2.37)$$

(ii) *Suppose*

$$A_0(t, s) \in \mathbb{R}_{*+}^{n \times n}, \quad A_1(s) \in \mathbb{R}_d^{n \times n}, \quad \text{a.e. } (t, s) \in \Delta^*, \text{ a.s.} \quad (2.38)$$

Moreover, there exists a continuous nondecreasing function $\rho : [0, T] \rightarrow [0, \infty)$ with $\rho(0) = 0$ such that

$$|A_0(t, s) - A_0(t', s)| \leq \rho(|t - t'|), \quad t, t' \in [0, T], \quad s \in [0, t \wedge t'], \text{ a.s.}, \quad (2.39)$$

and

$$A_0(\tau, s) - A_0(t, s) \in \mathbb{R}_+^{n \times n}, \quad \forall 0 \leq s \leq t \leq \tau \leq T, \text{ a.s.} \quad (2.40)$$

Then for any $\varphi(\cdot) \in C_{\mathbb{F}}([0, T]; L^2(\Omega, \mathbb{R}^n))$, with

$$\varphi(\tau) \geq \varphi(t) \geq 0, \quad \forall 0 \leq s \leq t \leq \tau \leq T, \text{ a.s.}, \quad (2.41)$$

(2.35) admits a unique solution $X(\cdot) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; \mathbb{R}^n))$ and it satisfies:

$$X(t) \geq 0, \quad t \in [0, T], \text{ a.s.} \quad (2.42)$$

Note that between the above (i) and (ii), none of them includes the other. Condition (2.36) implies that the map $y \mapsto A_0(t, s)y$ is nondecreasing (for $y \geq 0$); whereas, condition (2.40) implies that the map $t \mapsto A_0(t, s)y$ is nondecreasing. The monotonicity of $\varphi(\cdot)$ is assumed in (ii), which is not needed in (i). We

will encounter a similar situation for BSVIEs a little later. Also, because of Example 2.6, $\mathbb{R}_+^{n \times n}$ in (2.36) cannot be replaced by $\mathbb{R}_{*+}^{n \times n}$.

Proof. (i) Define

$$(\mathcal{A}X)(t) = \int_0^t A_0(t, s)X(s)ds, \quad t \in [0, T].$$

By our condition, making use of Proposition 2.1, we see that

$$(\mathcal{A}X)(\cdot) \geq 0, \quad \forall X(\cdot) \in L_{\mathbb{F}}^2(0, T; \mathbb{R}^n), \quad X(\cdot) \geq 0.$$

Now, we define the following Picard iteration sequence

$$X^0(\cdot) = \varphi(\cdot), \quad X^k(\cdot) = \varphi(\cdot) + (\mathcal{A}X^{k-1})(\cdot), \quad k \geq 1.$$

By induction, it is easy to see that

$$X^k(\cdot) \geq \varphi(\cdot), \quad \forall k \geq 0.$$

Further,

$$\lim_{k \rightarrow \infty} \|X^k(\cdot) - X(\cdot)\|_{L_{\mathbb{F}}^2(0, T; \mathbb{R}^n)} = 0,$$

with $X(\cdot)$ being the solution to (2.35). Then it is easy to see that (2.37) holds.

(ii) Let $\Pi = \{\tau_k, 0 \leq k \leq N\}$ be an arbitrary set of finitely many \mathbb{F} -stopping times with $0 = \tau_0 < \tau_1 < \dots < \tau_N = T$, and we define its mesh size by

$$\|\Pi\| = \text{esssup}_{\omega \in \Omega} \max_{1 \leq k \leq N} |\tau_k - \tau_{k-1}|.$$

Let

$$A_0^\Pi(t, s) = \sum_{k=0}^{N-1} A_0(\tau_k, s)I_{[\tau_k, \tau_{k+1})}(t), \quad \varphi^\Pi(t) = \sum_{k=0}^{N-1} \varphi(\tau_k)I_{[\tau_k, \tau_{k+1})}(t).$$

Clearly, each $A_0(\tau_k, \cdot)$ is an \mathbb{F} -adapted bounded process, and each $\varphi(\tau_k)$ is an \mathcal{F}_{τ_k} -measurable random variable. Moreover, for each $k \geq 0$,

$$A_0(\tau_k, s) \in \mathbb{R}_{*+}^{n \times n}, \quad s \in [\tau_k, \tau_{k+1}), \quad \text{a.s.}, \quad (2.43)$$

and

$$0 \leq \varphi(\tau_k) \leq \varphi(\tau_{k+1}), \quad \text{a.s.} \quad (2.44)$$

Further,

$$|A_0^\Pi(t, s) - A_0(t, s)| = \sum_{k=0}^{N-1} |A_0(\tau_k, s) - A_0(t, s)|I_{[\tau_k, \tau_{k+1})}(t) \leq \sum_{k=0}^{N-1} \rho(t - \tau_k)I_{[\tau_k, \tau_{k+1})}(t) \leq \rho(\|\Pi\|).$$

Now, we let $X^\Pi(\cdot)$ be the solution to the following FSVIE:

$$X^\Pi(t) = \varphi^\Pi(t) + \int_0^t A_0^\Pi(t, s)X^\Pi(s)ds + \int_0^t A_1(s)X^\Pi(s)dW(s), \quad t \in [0, T]. \quad (2.45)$$

Then we can show that

$$\lim_{\|\Pi\| \rightarrow 0} \mathbb{E} \left[\sup_{t \in [0, T]} |X^\Pi(t) - X(t)|^2 \right] = 0. \quad (2.46)$$

We now want to show that

$$X^\Pi(t) \geq 0, \quad t \in [0, T], \quad \text{a.s.}, \quad (2.47)$$

which, together with (2.46) will lead to (2.42). To show (2.47), we look at $X^\Pi(\cdot)$ on each interval $[\tau_k, \tau_{k+1})$, $k = 0, 1, \dots, N-1$. First, on interval $[0, \tau_1)$, we have

$$X^\Pi(t) = \varphi(0) + \int_0^t A_0(0, s)X^\Pi(s)ds + \int_0^t A_1(s)X^\Pi(s)dW(s),$$

which is an FSDE, and $X^\Pi(\cdot)$ has continuous paths (on $[0, \tau_1)$). From Proposition 2.2, we have

$$X^\Pi(t) \geq 0, \quad t \in [0, \tau_1), \text{ a.s.}$$

In particular,

$$X^\Pi(\tau_1 - 0) = \varphi(0) + \int_0^{\tau_1} A_0(0, s)X^\Pi(s)ds + \int_0^{\tau_1} A_1(s)X^\Pi(s)dW(s) \geq 0. \quad (2.48)$$

Next, on $[\tau_1, \tau_2)$, we have (making use of (2.48))

$$\begin{aligned} X^\Pi(t) &= \varphi(\tau_1) + \int_0^{\tau_1} A_0(\tau_1, s)X^\Pi(s)ds + \int_0^{\tau_1} A_1(s)X^\Pi(s)dW(s) \\ &\quad + \int_{\tau_1}^t A_0(\tau_1, s)X^\Pi(s)ds + \int_{\tau_1}^t A_1(s)X^\Pi(s)dW(s) \\ &= \varphi(\tau_1) - \varphi(0) + X^\Pi(\tau_1 - 0) + \int_0^{\tau_1} (A_0(\tau_1, s) - A_0(0, s))X^\Pi(s)ds \\ &\quad + \int_{\tau_1}^t (A_0(\tau_1, s)X^\Pi(s))ds + \int_{\tau_1}^t A_1(s)X^\Pi(s)dW(s) \\ &\equiv \tilde{X}(\tau_1) + \int_{\tau_1}^t A_0(\tau_1, s)X^\Pi(s)ds + \int_{\tau_1}^t A_1(s)X^\Pi(s)dW(s), \end{aligned}$$

where, by (2.41) and (2.48),

$$\tilde{X}(\tau_1) \equiv \varphi(\tau_1) - \varphi(0) + X^\Pi(\tau_1 - 0) + \int_0^{\tau_1} (A_0(\tau_1, s) - A_0(0, s))X^\Pi(s)ds \geq 0.$$

Hence, one obtains

$$X^\Pi(t) \geq 0, \quad t \in [\tau_1, \tau_2).$$

By induction, we obtain (2.47). □

Based on the above result, it is not very hard for us to present comparison theorems for nonlinear FSVIEs. We prefer not to give the details here. One can cook up that by following the relevant details for BSVIEs which will be presented in the following section. To conclude this section, we present an example showing that in the case $A_1(\cdot) \neq 0$, as long as $t \mapsto A_0(t, \cdot)$ is not nondecreasing in the sense of (2.40), even if $A_0(t, s) \in \mathbb{R}_+^{n \times n}$, comparison theorem might still fail as well.

Example 2.10. Consider the following FSVIE:

$$X(t) = 1 + \int_0^t I_{[0, \tau]}(t)X(s)ds + \int_0^t X(s)dW(s), \quad t \in [0, T],$$

where $\tau \in (0, T)$. Clearly, the above is a special case of (2.30) with

$$\varphi(t) = 1, \quad A_0(t, s) = I_{[0, \tau]}(t), \quad A_1(s) = 1.$$

Thus, $t \mapsto A_0(t, s)$ is not nondecreasing. Let us solve this FSVIE. On $[0, \tau)$, we have

$$X(t) = 1 + \int_0^t X(s)ds + \int_0^t X(s)dW(s),$$

which is equivalent to the following:

$$dX(t) = X(t)dt + X(t)dW(t), \quad X(0) = 1.$$

Hence,

$$X(t) = e^{\frac{t}{2} + W(t)}, \quad t \in [0, \tau].$$

On $[\tau, T]$, we have

$$\begin{aligned} X(t) &= 1 + \int_0^t X(s)dW(s) = 1 + \int_0^\tau X(s)dW(s) + \int_\tau^t X(s)dW(s) \\ &= X(\tau - 0) - \int_0^\tau X(s)ds + \int_\tau^t X(s)dW(s), \end{aligned}$$

which is equivalent to the following:

$$dX(t) = X(t)dW(t), \quad X(\tau + 0) = X(\tau - 0) - \int_0^\tau X(s)ds.$$

Hence,

$$\begin{aligned} X(t) &= e^{-\frac{t-\tau}{2} + W(t) - W(\tau)} \left[X(\tau - 0) - \int_0^\tau X(s)ds \right] \\ &= e^{-\frac{t-\tau}{2} + W(t) - W(\tau)} \left[e^{\frac{\tau}{2} + W(\tau)} - \int_0^\tau e^{\frac{s}{2} + W(s)} ds \right]. \end{aligned}$$

Then $X(t) < 0$ for $t \in [\tau, T]$ if and only if

$$e^{\frac{\tau}{2} + W(\tau)} < \int_0^\tau e^{\frac{s}{2} + W(s)} ds.$$

By convexity of $\lambda \mapsto e^\lambda$, we have

$$\frac{1}{\tau} \int_0^\tau e^{\frac{s}{2} + W(s)} ds \geq e^{\frac{1}{\tau} \int_0^\tau [\frac{s}{2} + W(s)] ds}.$$

Hence, $X(t) < 0$ for some $t \in [\tau, T]$ is implied by

$$e^{\frac{\tau}{2} + W(\tau)} < \tau e^{\frac{1}{\tau} \int_0^\tau (\frac{s}{2} + W(s)) ds},$$

which is equivalent to

$$\begin{aligned} \frac{\tau}{2} + W(\tau) &< \ln \tau + \frac{1}{\tau} \int_0^\tau \left(\frac{s}{2} + W(s) \right) ds = \ln \tau + \frac{\tau}{4} + \frac{1}{\tau} \int_0^\tau W(s) ds \\ &= \ln \tau + \frac{\tau}{4} + \frac{1}{\tau} sW(s) \Big|_0^\tau - \int_0^\tau s dW(s) = \ln \tau + \frac{\tau}{4} + W(\tau) - \int_0^\tau s dW(s). \end{aligned}$$

This is further equivalent to the following:

$$\int_0^\tau s dW(s) < \ln \tau - \frac{\tau}{4}.$$

The left hand side of the above is a normal random variable. Hence,

$$\mathbb{P} \left(\int_0^\tau s dW(s) < \ln \tau - \frac{\tau}{4} \right) > 0,$$

which implies

$$\mathbb{P} \left(X(t) < 0 \right) > 0, \quad t \in (\tau, T].$$

Although in the above, $A_0(\cdot, \cdot)$ is discontinuous, it is not hard for us to replace it by a continuous one and still have the same conclusion.

3 Comparison Theorems for BSVIEs.

In this section we consider various comparison theorems for BSVIEs.

3.1 Comparison for adapted solutions.

We first consider the following type BSVIEs: For $i = 0, 1$,

$$Y^i(t) = \psi^i(t) + \int_t^T g^i(t, s, Y^i(s), Z^i(t, s))ds - \int_t^T Z^i(t, s)dW(s), \quad t \in [0, T]. \quad (3.1)$$

The key feature here is that the generator $g^i(\cdot)$ is independent of $Z^i(s, t)$. For any adapted solution $(Y^i(\cdot), Z^i(\cdot, \cdot)) \in \mathcal{H}_\Delta^p[0, T]$ of the above, we need only the values $Z^i(t, s)$ of $Z^i(\cdot, \cdot)$ for $0 \leq t \leq s \leq T$, and the values $Z^i(t, s)$ of $Z^i(\cdot, \cdot)$ for $0 \leq s < t \leq T$ are irrelevant. Consequently, the notion of M-solution is not necessary for BSVIE of form (3.1). For the generator $g(\cdot)$ of BSVIE (3.1), we adopt the following assumption.

(BV1) Let $g^i : \Delta \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ be measurable such that $s \mapsto g^i(t, s, y, z)$ is \mathbb{F} -progressively measurable, $(y, z) \mapsto g^i(t, s, y, z)$ is uniformly Lipschitz, $(t, s) \mapsto g^i(t, s, 0, 0)$ is uniformly bounded.

It is known that under (BV1), for any $\psi^i(\cdot) \in C_{\mathcal{F}_T}([0, T]; L^2(\Omega; \mathbb{R}^n))$, BSVIE (3.1) admits a unique adapted solution $(Y^i(\cdot), Z^i(\cdot, \cdot)) \in \mathcal{H}_\Delta^2[0, T]$. We want to look at if a proper comparison between $Y^1(\cdot)$ and $Y^0(\cdot)$ holds under certain additional conditions on $g^i(\cdot)$ and $\psi^i(\cdot)$. To begin with, let us first look at the following simple BSVIEs: For $i = 0, 1$,

$$Y^i(t) = \psi^i(t) + \int_t^T g^i(t, s, Z^i(t, s))ds - \int_t^T Z^i(t, s)dW(s), \quad t \in [0, T], \quad (3.2)$$

with the generators $g^i(\cdot)$ are independent of $Y^i(s)$. We have the following result.

Proposition 3.1. *For $i = 0, 1$, let $g^i : \Delta \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ satisfy (BV1). Moreover,*

$$g^0(t, s, z) \leq g^1(t, s, z), \quad \forall (t, s, z) \in \Delta \times \mathbb{R}^n, \text{ a.s.} \quad (3.3)$$

and for either $i = 0$ or $i = 1$, $g_z^i(t, s, z)$ exists and

$$g_z^i(t, s, z) \in \mathbb{R}_d^{n \times n}, \quad (t, s, z) \in \Delta \times \mathbb{R}^n, \text{ a.s.} \quad (3.4)$$

Then the adapted solutions $(Y^i(\cdot), Z^i(\cdot, \cdot)) \in \mathcal{H}_\Delta^2[0, T]$ of BSVIE (3.2) with

$$\psi^0(t) \leq \psi^1(t), \quad t \in [0, T], \text{ a.s.}, \quad (3.5)$$

satisfies

$$Y^0(t) \leq Y^1(t), \quad t \in [0, T], \text{ a.s.} \quad (3.6)$$

Proof. Fixed $t \in [0, T]$. For $i = 0, 1$, let $(\lambda^i(t, \cdot), \mu^i(t, \cdot))$ be the adapted solution to the following BSDE:

$$\lambda^i(t, r) = \psi^i(t) + \int_r^T g^i(t, s, \mu^i(t, s))ds - \int_r^T \mu^i(t, s)dW(s), \quad r \in [t, T].$$

By Theorem 2.5, we have that

$$\lambda^0(t, r) \leq \lambda^1(t, r), \quad r \in [t, T], \text{ a.s.} \quad (3.7)$$

By setting

$$Y^i(t) = \lambda^i(t, t), \quad Z^i(t, s) = \mu^i(t, s), \quad \forall (t, s) \in \Delta, \quad (3.8)$$

we see that $(Y^i(\cdot), Z^i(\cdot, \cdot))$ is the adapted solution to the BSVIE (3.2). Then (3.6) follows from (3.7). \square

Returning to BSVIEs (3.1), we have the following result.

Theorem 3.2. *Let (BV1) hold. Suppose $\bar{g} : \Delta \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ is measurable, $s \mapsto \bar{g}(t, s, y, z)$ is \mathbb{F} -progressively measurable, $(y, z) \mapsto \bar{g}(t, s, y, z)$ is uniformly Lipschitz, $y \mapsto \bar{g}(t, s, y, z)$ is nondecreasing, such that*

$$g^0(t, s, y, z) \leq \bar{g}(t, s, y, z) \leq g^1(t, s, y, z), \quad (t, s, y, z) \in \Delta \times \mathbb{R}^n \times \mathbb{R}^n, \text{ a.s.} \quad (3.9)$$

Moreover, $\bar{g}_z(t, s, y, z)$ exists and

$$\bar{g}_z(t, s, y, z) \in \mathbb{R}_d^{n \times n}, \quad (t, s, y, z) \in \Delta \times \mathbb{R}^n \times \mathbb{R}^n, \text{ a.s.} \quad (3.10)$$

Then for any $\psi^i(\cdot) \in C_{\mathcal{F}_T}([0, T]; L^2(\Omega; \mathbb{R}^n))$ satisfying

$$\psi^0(t) \leq \psi^1(t), \quad t \in [0, T], \text{ a.s.}, \quad (3.11)$$

the corresponding unique adapted solution $(Y^i(\cdot), Z^i(\cdot, \cdot)) \in \mathcal{H}_\Delta^2[0, T]$ of BSVIE (3.1) satisfy

$$Y^0(t) \leq Y^1(t), \quad t \in [0, T], \text{ a.s.} \quad (3.12)$$

Proof. Let $\bar{\psi}(\cdot) \in C_{\mathcal{F}_T}([0, T]; L^2(\Omega; \mathbb{R}^n))$ such that

$$\psi^0(t) \leq \bar{\psi}(t) \leq \psi^1(t), \quad t \in [0, T], \text{ a.s.}$$

Let $(\bar{Y}(\cdot), \bar{Z}(\cdot, \cdot))$ be the adapted solution to the following:

$$\bar{Y}(t) = \bar{\psi}(t) + \int_t^T \bar{g}(t, s, \bar{Y}(s), \bar{Z}(t, s))ds - \int_t^T \bar{Z}(t, s)dW(s), \quad t \in [0, T].$$

Set $\tilde{Y}_0(\cdot) = Y^0(\cdot)$ and consider the following BSVIE:

$$\tilde{Y}_1(t) = \bar{\psi}(t) + \int_t^T \bar{g}(t, s, \tilde{Y}_0(s), \tilde{Z}_1(t, s))ds - \int_t^T \tilde{Z}_1(t, s)dW(s), \quad t \in [0, T].$$

Let $(\tilde{Y}_1(\cdot), \tilde{Z}_1(\cdot, \cdot)) \in \mathcal{H}_\Delta^2[0, T]$ be the unique adapted solution to the above. Since

$$\begin{cases} \bar{g}(t, s, \tilde{Y}_0(s), z) \leq g^1(t, s, \tilde{Y}_0(s), z), & (t, s, z) \in \Delta \times \mathbb{R}^n, \text{ a.s.}, \\ \bar{g}_z(t, s, \tilde{Y}_0(s), z) \in \mathbb{R}_d^{n \times n}, & (t, s, z) \in \Delta \times \mathbb{R}^n, \text{ a.s.}, \\ \bar{\psi}(t) \leq \psi^1(t), & t \in [0, T], \text{ a.s.} \end{cases}$$

By Proposition 3.1, we obtain that

$$\tilde{Y}_1(t) \leq \tilde{Y}_0(t), \quad t \in [0, T].$$

Next, we consider the following BSVIE:

$$\tilde{Y}_2(t) = \bar{\psi}(t) + \int_t^T \bar{g}(t, s, \tilde{Y}_1(s), \tilde{Z}_2(t, s))ds - \int_t^T \tilde{Z}_2(t, s)dW(s), \quad t \in [0, T],$$

and let $(\tilde{Y}_2(\cdot), \tilde{Z}_2(\cdot, \cdot)) \in \mathcal{H}_\Delta^2[0, T]$ be the adapted solution to the above. Now, since $y \mapsto \bar{g}(t, s, y, z)$ is nondecreasing, we have

$$\bar{g}(t, s, \tilde{Y}_1(s), z) \leq \bar{g}(t, s, \tilde{Y}_0(s), z), \quad \forall (t, s, z) \in \Delta \times \mathbb{R}^n.$$

Hence, similar to the above, we obtain

$$\tilde{Y}_2(t) \leq \tilde{Y}_1(t), \quad t \in [0, T], \text{ a.s.}$$

By induction, we can construct a sequence $\{(\tilde{Y}_k(\cdot), \tilde{Z}_k(\cdot, \cdot))\}_{k \geq 1} \subseteq \mathcal{H}_\Delta^2[0, T]$ such that

$$\tilde{Y}_k(t) = \bar{\psi}(t) + \int_t^T \bar{g}(t, s, \tilde{Y}_{k-1}(s), \tilde{Z}_k(t, s)) ds - \int_t^T \tilde{Z}_k(t, s) dW(s), \quad t \in [0, T],$$

and

$$Y^1(t) = \tilde{Y}_0(t) \geq \tilde{Y}_1(t) \geq \tilde{Y}_2(t) \cdots, \quad t \in [0, T], \text{ a.s.} \quad (3.13)$$

Next we will show that the sequence $\{(\tilde{Y}_k(\cdot), \tilde{Z}_k(\cdot, \cdot))\}_{k \geq 1}$ is Cauchy in $\mathcal{H}_\Delta^2[0, T]$. To show this, we introduce an equivalent norm of $\mathcal{H}_\Delta^2[0, T]$ as

$$\|(y(\cdot), z(\cdot, \cdot))\|_{\mathcal{H}_\Delta^2[0, T]}^2 = \mathbb{E} \int_0^T e^{\beta t} |y(t)|^2 dt + \mathbb{E} \int_0^T e^{\beta t} \int_t^T |z(t, s)|^2 ds dt,$$

with $(y(\cdot), z(\cdot, \cdot)) \in \mathcal{H}_\Delta^2[0, T]$, and β being a constant undetermined. By utilizing a stability estimate in [25], we have

$$\begin{aligned} & \mathbb{E} |\tilde{Y}_k(t) - \tilde{Y}_\ell(t)|^2 + \mathbb{E} \int_t^T |\tilde{Z}_k(t, s) - \tilde{Z}_\ell(t, s)|^2 ds \\ & \leq K \mathbb{E} \left(\int_t^T |\bar{g}(t, s, \tilde{Y}_{k-1}(s), \tilde{Z}_k(t, s)) - \bar{g}(t, s, \tilde{Y}_{\ell-1}(s), \tilde{Z}_k(t, s))| ds \right)^2. \end{aligned} \quad (3.14)$$

Consequently, we arrive at

$$\begin{aligned} & \mathbb{E} \int_0^T e^{\beta t} |\tilde{Y}_k(t) - \tilde{Y}_\ell(t)|^2 dt + \mathbb{E} \int_0^T e^{\beta t} \left(\int_t^T |\tilde{Z}_k(t, s) - \tilde{Z}_\ell(t, s)|^2 ds \right) dt \\ & \leq K \mathbb{E} \int_0^T e^{\beta t} \left(\int_t^T |\bar{g}(t, s, \tilde{Y}_{k-1}(s), \tilde{Z}_k(t, s)) - \bar{g}(t, s, \tilde{Y}_{\ell-1}(s), \tilde{Z}_k(t, s))| ds \right)^2 dt \\ & \leq K \mathbb{E} \int_0^T e^{\beta t} \left(\int_t^T |\tilde{Y}_{k-1}(s) - \tilde{Y}_{\ell-1}(s)| ds \right)^2 dt \\ & \leq K \mathbb{E} \int_0^T |\tilde{Y}_{k-1}(s) - \tilde{Y}_{\ell-1}(s)|^2 ds \int_0^s e^{\beta t} dt \leq \frac{K}{\beta} \mathbb{E} \int_0^T e^{\beta s} |\tilde{Y}_{k-1}(s) - \tilde{Y}_{\ell-1}(s)|^2 ds. \end{aligned} \quad (3.15)$$

Note that the constant $K > 0$ in the above can be chosen independent of $\beta > 0$. Thus by choosing a β such that $\frac{K}{\beta} < 1$, we obtain immediately that $\{(\tilde{Y}_k(\cdot), \tilde{Z}_k(\cdot, \cdot))\}_{k \geq 1}$ is Cauchy in $\mathcal{H}_\Delta^2[0, T]$. Hence, there exists a $(\tilde{Y}(\cdot), \tilde{Z}(\cdot, \cdot)) \in \mathcal{H}_\Delta^2[0, T]$ such that

$$\lim_{k \rightarrow \infty} \left[\mathbb{E} \int_0^T |\tilde{Y}^k(t) - \tilde{Y}(t)|^2 dt + \mathbb{E} \int_0^T e^{\beta t} \left(\int_t^T |\tilde{Z}^k(t, s) - \tilde{Z}(t, s)|^2 ds \right) dt \right] = 0,$$

and

$$\tilde{Y}(t) = \bar{\psi}(t) + \int_t^T \bar{g}(t, s, \tilde{Y}(s), \tilde{Z}(t, s)) ds - \int_t^T \tilde{Z}(t, s) dW(s), \quad t \in [0, T].$$

By uniqueness, we have

$$\bar{Y}(t) = \tilde{Y}(t) \leq \tilde{Y}_0(t) = Y^1(t), \quad t \in [0, T], \text{ a.s.}$$

Similarly, we can prove that

$$Y^0(t) \leq \bar{Y}(t), \quad t \in [0, T], \text{ a.s.}$$

Therefore, our conclusion follows. \square

It is easy to cook up an example for which $y \mapsto g^i(t, s, y, z)$ is not nondecreasing for $i = 0, 1$, but a $\bar{g}(\cdot)$ satisfying conditions of Theorem 3.2 can be constructed. For example,

$$g^0(t, s, y, z) \equiv \sin y \leq 1 \equiv \bar{g}(t, s, y, z) \leq 2 + \cos y \equiv g^1(t, s, y, z).$$

Condition (3.9) means that in the tube

$$\left\{ \left[g^0(t, s, y, z), g^1(t, s, y, z) \right] \mid (t, s, y, z) \in \Delta \times \mathbb{R}^n \times \mathbb{R}^n \right\},$$

there exists a selection $\bar{g}(t, s, y, z)$ which is nondecreasing in y , and (3.10) is satisfied. Therefore, the condition assumed in Theorem 3.2 is a kind of *generalized nondecreasing condition* for the maps $y \mapsto g^i(t, s, y, z)$, although these maps themselves are not necessarily nondecreasing. Consequently, it is expected that condition (3.9) excludes many other situations. To see that, let us look at two examples.

Example 3.3. Consider one-dimensional linear BSVIE

$$Y(t) = t - \int_t^T Y(s)ds - \int_t^T Z(t, s)dW(s), \quad t \in [0, T].$$

It is clear that if $(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}_\Delta^2[0, T]$ is the adapted solution, then $Z(\cdot, \cdot) = 0$ and

$$Y(t) = e^{t-T}(T+1) - 1, \quad t \in [0, T].$$

Consequently,

$$Y(t) < 0, \quad t \in [0, T - \ln(T+1)].$$

Therefore, comparison theorem fails for this example. This example corresponds to the case

$$g^i(t, s, y, z) = -y, \quad i = 0, 1, \quad \psi^1(t) = t, \quad \psi^0(t) = 0.$$

Apparently, $\bar{g}(\cdot)$ satisfying the conditions in Theorem 3.2 does not exist.

Example 3.4. Consider

$$Y(t) = 1 + \int_t^T (t-1)Y(s)ds - \int_t^T Z(t, s)dW(s), \quad t \in [0, T].$$

Again, if $(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}_\Delta^2[0, T]$ the adapted solution, then $Z(\cdot, \cdot) = 0$. Now, we denote

$$y(t) = \int_t^T Y(s)ds.$$

Then

$$\dot{y}(t) = -Y(t) = -1 - (t-1) \int_t^T Y(s)ds = -1 - (t-1)y(t).$$

Hence,

$$0 = y(T) = e^{-\int_t^T (s-1)ds} y(t) - \int_t^T e^{-\int_\tau^T (s-1)ds} d\tau.$$

This yields

$$y(t) = \int_t^T e^{\int_t^\tau (s-1)ds} d\tau = \int_t^T e^{\frac{1}{2}[(\tau-1)^2 - (t-1)^2]} d\tau.$$

Therefore,

$$Y(t) = 1 + (t-1)y(t) = 1 + (t-1) \int_t^T e^{\frac{1}{2}[\tau^2 - t^2 - 2(\tau-t)]} d\tau.$$

Consequently,

$$Y(0) = 1 - \int_0^T e^{\frac{1}{2}\tau^2 - \tau} d\tau < 0,$$

provided $T > 0$ is large. Thus, comparison theorem fails for this example as well. This example corresponds to the case

$$g^i(t, s, y, z) = (t-1)y, \quad i = 0, 1, \quad \psi^1(t) = 1, \quad \psi^0(t) = 0.$$

Again, for this example, the generator $\bar{g}(\cdot)$ satisfying the conditions in Theorem 3.2 does not exist.

Let us take a closer look at the above two examples. In Example 3.3, $t \mapsto \psi^1(t)$ is increasing, and in Example 3.4, $t \mapsto g^i(t, s, y, z)$ is increasing for $y > 0$. In a certain sense, these conditions actually prevent the comparison theorem from being true for these examples. On the other hand, we keep in mind that when $\psi(t)$ and $g^i(t, s, y, z)$ are independent of t , the above two situations do not appear. Hence, it is natural to ask if comparison theorem remains when $\psi(t)$ and $g^i(t, s, y, z)$ do depend on t , and the generalized nondecreasing condition (3.9) is not assumed. The answer is positive. Before we state and prove a general positive result, let us look at the following example.

Example 3.5. Consider the following BSVIE:

$$Y(t) = \int_t^T [s - t - Y(s)] ds - \int_t^T Z(t, s) dW(s), \quad t \in [0, T].$$

In this case, we have

$$\psi(t) \equiv 0, \quad g(t, s, y, z) = s - t - y.$$

Thus, condition of Theorem 3.2 fails. However, it is easy to check that the unique adapted solution $(Y(\cdot), Z(\cdot, \cdot))$ is given by

$$Y(s) = e^{s-T} + T - s - 1, \quad Z(t, s) = 0.$$

is the unique solution here. Clearly,

$$Y(s) \geq 0, \quad s \in [0, T],$$

comparison theorem holds. Note that in this case, $t \mapsto g(t, s, y, z)$ is nondecreasing. On the other hand, the BSVIE is equivalent to the following:

$$Y(t) = \frac{(T-t)^2}{2} - \int_t^T Y(s) ds - \int_t^T Z(t, s) dW(s), \quad t \in [0, T],$$

with

$$\psi(t) = \frac{(T-t)^2}{2}, \quad g(t, s, y) = -y.$$

For this, we have that $t \mapsto \psi(t)$ is non-increasing.

Inspired by the above example, we see that without condition (3.9), one might still have comparison theorem. We now establish such kind of results. Let us begin with a result for linear BSVIEs. More precisely, we consider the following linear BSVIE:

$$Y(t) = \psi(t) + \int_t^T [A(t, s)Y(s) + B(s)Z(t, s)] ds - \int_t^T Z(t, s) dW(s), \quad t \in [0, T]. \quad (3.16)$$

Note that the coefficient $B(s)$ of $Z(t, s)$ is independent of t . We have the following theorem.

Theorem 3.6. Let $A : \Delta \times \Omega \rightarrow \mathbb{R}^{n \times n}$ and $B : [0, T] \times \Omega \rightarrow \mathbb{R}^{n \times n}$ be uniformly bounded, with $B(\cdot)$ being \mathbb{F} -progressively measurable, for each $t \in [0, T]$, $s \mapsto A(t, s)$ being \mathbb{F} -progressively measurable, and for each $s \in [0, T]$, $t \mapsto A(t, s)$ being continuous. Moreover,

$$A(t, s) \in \mathbb{R}_{*+}^{n \times n}, \quad (t, s) \in \Delta, \text{ a.s. }, \quad (3.17)$$

$$A(t, s) - A(\tau, s) \in \mathbb{R}_+^{n \times n}, \quad 0 \leq t \leq \tau \leq s \leq T, \text{ a.s. }, \quad (3.18)$$

$$B(s) \in \mathbb{R}_d^{n \times n}, \quad s \in [0, T], \text{ a.s. } \quad (3.19)$$

Then for any $\psi(\cdot) \in C_{\mathcal{F}_T}([0, T]; L^2(\Omega; \mathbb{R}^n))$ with

$$\psi(t) \geq \psi(s) \geq 0, \quad 0 \leq t \leq s \leq T, \text{ a.s. }, \quad (3.20)$$

the adapted solution $Y(\cdot), Z(\cdot, \cdot)$ of linear BSVIE (3.16) satisfies the following:

$$Y(t) \geq 0, \quad t \in [0, T], \text{ a.s.} \quad (3.21)$$

We point out that $A(t, s)$ satisfying (3.17) (which is always true if $n = 1$) is not necessarily in $\mathbb{R}_+^{n \times n}$. Therefore, the map $y \mapsto A(t, s)y$ is not necessarily nondecreasing. Also, when $A(t, s)$ is independent of t , (3.18) is automatically true.

Proof. Let

$$A(t, s) = \sum_{k=1}^N A_k(s) I_{(t_{k-1}, t_k]}(t), \quad \psi(t) = \sum_{k=1}^N \psi_k I_{(t_{k-1}, t_k]}(t),$$

where $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$ is a partition of $[0, T]$, and each $A_k(\cdot)$ is an \mathbb{F} -adapted process valued in $\mathbb{R}^{n \times n}$,

$$A_k(s) \in \mathbb{R}_{*+}^{n \times n}, \quad s \in [0, T], \text{ a.s.}$$

$$A_{k-1}(s) - A_k(s) \in \mathbb{R}_+^{n \times n}, \quad s \in [0, T], \quad k = 1, \dots, N, \text{ a.s.},$$

each ψ_k is an \mathcal{F}_T -measurable random variable valued in \mathbb{R}^n such that

$$\psi_1 \geq \psi_2 \geq \dots \geq \psi_{N-1} \geq \psi_N \geq 0, \quad \text{a.s.}$$

Let $(Y(\cdot), Z(\cdot, \cdot))$ be the adapted solution to the BSVIE. On $(t_{N-1}, t_N]$, we have

$$Y(t) = \psi_N + \int_t^T \left(A_N(s)Y(s) + B(s)Z(t, s) \right) ds - \int_t^T Z(t, s) dW(s).$$

By uniqueness of BSDEs, we see that

$$(Y(s), Z(t, s)) \equiv (Y_N(s), Z_N(s)), \quad \forall t_{N-1} < t \leq s \leq T,$$

with $(Y_N(\cdot), Z_N(\cdot))$ being the adapted solution to the following BSDE:

$$Y_N(t) = \psi_N + \int_t^T \left(A_N(s)Y_N(s) + B(s)Z_N(s) \right) ds - \int_t^T Z_N(s) dW(s), \quad t \in (t_{N-1}, t_N].$$

Further, under our condition, by Proposition 2.4, we have

$$Y(t) \equiv Y_N(t) \geq 0, \quad t \in (t_{N-1}, t_N], \text{ a.s.}$$

In particular,

$$Y(t_{N-1} + 0) = \psi_N + \int_{t_{N-1}}^T \left(A_N(s)Y(s) + B(s)Z_N(s) \right) ds - \int_{t_{N-1}}^T Z_N(s) dW(s) \geq 0, \quad \text{a.s.}$$

Next, for $t \in (t_{N-2}, t_{N-1}]$, we have

$$\begin{aligned} Y(t) &= \psi_{N-1} + \int_t^T \left(A_{N-1}(s)Y(s) + B(s)Z(t, s) \right) ds - \int_t^T Z(t, s) dW(s) \\ &= \psi_{N-1} - \psi_N + Y(t_{N-1} + 0) + \int_{t_{N-1}}^T [A_{N-1}(s) - A_N(s)] Y(s) ds \\ &\quad + \int_{t_{N-1}}^T B(s) [Z(t, s) - Z_N(s)] ds - \int_{t_{N-1}}^T [Z(t, s) - Z_N(s)] dW(s) \\ &\quad + \int_t^{t_{N-1}} \left(A_{N-1}(s)Y(s) + B(s)Z(t, s) \right) ds - \int_t^{t_{N-1}} Z(t, s) dW(s). \end{aligned}$$

Let $(\tilde{Y}_N(\cdot), \tilde{Z}_N(\cdot))$ be the adapted solution to the following BSDE:

$$\begin{aligned}\tilde{Y}_N(\tau) &= \psi_{N-1} - \psi_N + Y(t_{N-1} + 0) + \int_{t_{N-1}}^T [A_{N-1}(s) - A_N(s)] Y(s) ds \\ &\quad + \int_{\tau}^T B(s) \tilde{Z}_N(s) ds - \int_{\tau}^T \tilde{Z}_N(s) dW(s), \quad \tau \in (t_{N-1}, T].\end{aligned}$$

Since

$$\psi_{N-1} - \psi_N + Y(t_{N-1} + 0) + \int_{t_{N-1}}^T [A_{N-1}(s) - A_N(s)] Y(s) ds \geq 0,$$

by our conditions, using Proposition 2.4, we have

$$\tilde{Y}_N(\tau) \geq 0, \quad \tau \in (t_{N-1}, T], \text{ a.s.}$$

In particular,

$$\begin{aligned}\tilde{Y}_N(t_{N-1} + 0) &= \psi_{N-1} - \psi_N + Y(t_{N-1} + 0) + \int_{t_{N-1}}^T [A_{N-1}(s) - A_N(s)] Y(s) ds \\ &\quad + \int_{t_{N-1}}^T B(s) \tilde{Z}_N(s) ds - \int_{t_{N-1}}^T \tilde{Z}_N(s) dW(s) \geq 0, \quad \text{a.s.}\end{aligned}$$

On the other hand, by the uniqueness of adapted solutions to the above BSVIEs, it is necessary that

$$Z(t, s) = Z_N(s) + \tilde{Z}_N(s), \quad (t, s) \in (t_{N-2}, t_{N-1}] \times (t_{N-1}, t_N].$$

Then $\tilde{Y}_N(t_{N-1})$ is $\mathcal{F}_{t_{N-1}}$ -measurable, $s \mapsto Z(t, s)$ is \mathbb{F} -adapted, and for $t \in (t_{N-2}, t_{N-1}]$, and

$$Y(t) = \tilde{Y}_N(t_{N-1} + 0) + \int_t^{t_{N-1}} (A_{N-1}(s) Y(s) + B(s) Z(t, s)) ds - \int_t^{t_{N-1}} Z(t, s) dW(s).$$

Next, we let $(Y_{N-1}(\cdot), Z_{N-1}(\cdot))$ be the adapted solution to the following BSDE:

$$\begin{aligned}Y_{N-1}(t) &= \tilde{Y}(t_{N-1} + 0) + \int_t^{t_{N-1}} (A_{N-1}(s) Y_{N-1}(s) + B(s) Z_{N-1}(s)) ds \\ &\quad - \int_t^{t_{N-1}} Z_{N-1}(s) dW(s), \quad t \in [t_{N-2}, t_{N-1}].\end{aligned}$$

By uniqueness of adapted solutions to BSDEs, we must have

$$(Y(s), Z(t, s)) = (Y_{N-1}(s), Z_{N-1}(s)), \quad t \in (t_{N-2}, t_{N-1}], \quad s \in [t, t_{N-1}].$$

Also, by $\tilde{Y}(t_{N-1}) \geq 0$, we obtain

$$Y(t) \equiv Y_{N-1}(t) \geq 0, \quad t \in (t_{N-2}, t_{N-1}].$$

Therefore,

$$Y(t) \geq 0, \quad t \in (t_{N-2}, t_N], \text{ a.s.}$$

Then, by induction, we obtain

$$Y(t) \geq 0, \quad t \in [0, T].$$

Finally, by approximation, we obtain the general case. \square

In the above proof, the condition that the coefficient $B(s)$ of $Z(t, s)$ is independent of t is crucial. It is desired if the above remains true when $B(s)$ is replaced by $B(t, s)$. Unfortunately, we do not have a confirmative answer at the moment.

Having the above result, we now state a result for nonlinear case.

Theorem 3.7. *Let $g^i : \Delta \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ satisfy (BV1), and the following hold*

$$g^i(t, s, y, z) = h^i(t, s, y) + B(s)z, \quad (t, s, y, z) \in \Delta \times \mathbb{R}^n \times \mathbb{R}^n, \quad (3.22)$$

for some $h^i : \Delta \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ and $B(\cdot) \in L_{\mathbb{F}}^{\infty}(0, T; \mathbb{R}^{n \times n})$. Moreover,

$$\begin{aligned} h^1(t, s, y) - h^0(t, s, y) &\geq h^1(\tau, s, y) - h^0(\tau, s, y) \geq 0, \\ \forall y \in \mathbb{R}^n, \quad 0 \leq t \leq \tau \leq s \leq T, \quad \text{a.s.} \end{aligned} \quad (3.23)$$

and for either $i = 0$ or $i = 1$, $y \mapsto h^i(t, s, y)$ is differentiable with

$$h_y^i(t, s, y) \in \mathbb{R}_{*+}^{n \times n}, \quad h_y^i(t, s, y) - h_y^i(\tau, s, y) \in \mathbb{R}_{+}^{n \times n}, \quad 0 \leq t \leq \tau \leq s \leq T, \quad y \in \mathbb{R}^n, \quad \text{a.s.} \quad (3.24)$$

Then for any $\psi^i(\cdot) \in C_{\mathcal{F}_T}([0, T]; L^2(\Omega; \mathbb{R}^n))$ with

$$\psi^1(t) - \psi^0(t) \geq \psi^1(\tau) - \psi^0(\tau) \geq 0, \quad 0 \leq t \leq \tau \leq T, \quad \text{a.s.} \quad (3.25)$$

the corresponding adapted solutions $(Y^i(\cdot), Z^i(\cdot, \cdot))$ of BSVIEs (3.1) satisfy

$$Y^1(t) \geq Y^0(t), \quad t \in [0, T], \quad \text{a.s.}$$

Proof. Suppose that $y \mapsto h^0(t, s, y)$ is differentiable and (3.24) holds for $i = 0$. Then we have

$$\begin{aligned} Y^1(t) - Y^0(t) &= \psi^1(t) - \psi^0(t) + \int_t^T \left[h^1(t, s, Y^1(s)) - h^0(t, s, Y^1(s)) \right] ds \\ &\quad + \int_t^T \left[A(t, s) \left(Y^1(s) - Y^0(s) \right) + B(s) \left(Z^1(t, s) - Z^0(t, s) \right) \right] ds \\ &\quad - \int_t^T \left(Z^1(t, s) - Z^0(t, s) \right) dW(s), \end{aligned}$$

where

$$A(t, s) = \int_0^1 h_y^0(t, s, Y^0(s) + \beta[Y^1(s) - Y^0(s)]) d\beta, \quad (t, s) \in \Delta.$$

Then our conclusion follows from Theorem 3.6. □

Note that in the above theorem, we have not assumed any sort of nondecreasing conditions on $y \mapsto g^i(t, s, y, z)$. Also, when $h^i(t, s, y)$ are independent of t , condition (3.17) is reduced to

$$h^1(s, y) \geq h^0(s, y), \quad (s, y) \in [0, T] \times \mathbb{R}^n, \quad \text{a.s.} \quad ,$$

and condition (3.24) is automatically true. Finally, if $y \mapsto h^i(t, s, y)$ is just Lipschitz and not necessarily differentiable, we may modify condition (3.24) in a proper way so that the same conclusion remains. On the other hand, we have seen that our result does not fully recover the comparison theorem for general nonlinear n -dimensional BSDEs. At the moment, this is the best that we can do.

3.2 Comparison theorem for adapted M-solutions.

In this subsection, we discuss the following type BSVIEs:

$$Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(s, t)) ds - \int_t^T Z(t, s) dW(s), \quad t \in [0, T]. \quad (3.26)$$

Note that since the generator $g(\cdot)$ depends on $Z(s, t)$, the notion of adapted solution in $\mathcal{H}_\Delta^p[0, T]$ will not be enough. Therefore, we adopt the notion of adapted M-solution to the above BSVIE ([25]). More precisely, an adapted M-solution is an adapted solution $(Y(\cdot), Z(\cdot, \cdot))$ which belongs to $\mathcal{M}^p[0, T]$. The following is a standard assumption for the BSVIE (3.26).

(BV2) For $i = 0, 1$, the maps $g^i : \Delta \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ is measurable, $s \mapsto g^i(t, s, y, \zeta)$ is \mathbb{F} -progressively measurable, $(y, \zeta) \mapsto g^i(t, s, y, \zeta)$ is uniformly Lipschitz, $(t, s) \mapsto g^i(t, s, 0, 0)$ is uniformly bounded.

By [25], we know that under (BV2), for any $\psi(\cdot) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; \mathbb{R}^n))$, (3.26) admits a unique adapted M-solution $(Y(\cdot), Z(\cdot, \cdot))$. We will use a dual principle ([25]) to prove the comparison theorem for adapted M-solution. The results of this subsection also corrects relevant ones in [23, 24]. Before going further, let us look at a simple example.

Example 3.8. Consider the following one-dimensional BSVIE:

$$Y(t) = \psi(t) + \int_t^T \frac{2T-t}{2T-s} Z(s, t) ds - \int_t^T Z(t, s) dW(s), \quad t \in [0, T]. \quad (3.27)$$

We introduce the following FSVIE:

$$X(t) = 1 + \int_0^t \frac{2T-s}{2T-t} X(s) dW(s), \quad t \in [0, T], \quad (3.28)$$

which is the equation in Example 2.7, and

$$\mathbb{P}\{\omega; X(t, \omega) < 0\} > 0, \quad \forall t \in [0, T].$$

Hence by taking

$$\psi(t) = I_A(t, \omega), \quad A = \{(t, \omega); X(t, \omega) < 0\}, \quad t \in [0, T],$$

by the duality principle ([25]), we have

$$\mathbb{E} \int_0^T Y(t) dt = \mathbb{E} \int_0^T X(t) I_A(t, \omega) dt < 0,$$

which means that $Y(\cdot) \geq 0$ on $[0, T]$ could not be true, although $\psi(\cdot) \geq 0$.

The above example shows that comparison theorem may fail for linear BSVIEs if in the generator, the coefficient of $Z(s, t)$ depends both on t and s . The above example suggests us that if linear BSVIEs are considered for comparison of adapted M-solutions, the following should be a proper form:

$$Y(t) = \psi(t) + \int_t^T \left(A(t, s) Y(s) + C(t) Z(s, t) \right) ds - \int_t^T Z(t, s) dW(s), \quad t \in [0, T]. \quad (3.29)$$

Note that $Z(t, s)$ does not appear in the drift term, and the coefficient $C(t)$ of $Z(s, t)$ is independent of s . For such an equation, we have the following result, which is comparable with Theorem 3.6.

Theorem 3.9. Let $A : \Delta \times \Omega \rightarrow \mathbb{R}^{n \times n}$ and $C : [0, T] \times \Omega \rightarrow \mathbb{R}^{n \times n}$ be uniformly bounded, with $C(\cdot)$ being \mathbb{F} -progressively measurable, for each $t \in [0, T]$, $s \mapsto A(t, s)$ being \mathbb{F} -progressively measurable, and for each $s \in [0, T]$, $t \mapsto A(s, t)$ is continuous. Further,

$$A(t, s) \in \mathbb{R}_{*+}^{n \times n}, \quad (t, s) \in \Delta, \text{ a.s. }, \quad (3.30)$$

$$A(s, \tau) - A(s, t) \in \mathbb{R}_+^{n \times n}, \quad \forall s \leq t \leq \tau \leq T, \quad s \in [0, T], \text{ a.s. }, \quad (3.31)$$

$$C(t) \in \mathbb{R}_d^{n \times n}, \quad \text{a.e. } t \in [0, T], \text{ a.s. } \quad (3.32)$$

Then the adapted M -solution $(Y(\cdot), Z(\cdot, \cdot))$ of linear BSVIE (3.29) with $\psi(\cdot) \in C_{\mathcal{F}_T}(0, T; L^2(\Omega; \mathbb{R}^n))$, $\psi(\cdot) \geq 0$ satisfies

$$\mathbb{E}_t \int_t^T Y(s) ds \geq 0, \quad \forall t \in [0, T], \text{ a.s.} \quad (3.33)$$

Proof. Pick any $\eta(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$ with $\eta(\cdot) \geq 0$, consider the following linear FSVIE:

$$X(t) = \varphi(t) + \int_0^t A(s, t)^T X(s) ds + \int_0^t C(s)^T X(s) dW(s), \quad t \in [0, T], \quad (3.34)$$

with

$$\varphi(t) = \int_0^t \eta(s) ds, \quad t \in [0, T].$$

By our conditions on $A(\cdot, \cdot)$ and $C(\cdot)$, using Proposition 2.7, we have

$$X(t) \geq 0, \quad t \in [0, T], \text{ a.s.}$$

Then by duality theorem ([25]), one obtains

$$\begin{aligned} 0 &\leq \mathbb{E} \int_0^T \langle \psi(t), X(t) \rangle dt = \mathbb{E} \int_0^T \langle \varphi(t), Y(t) \rangle dt \\ &= \mathbb{E} \int_0^T \int_0^t \langle \eta(s), Y(t) \rangle ds dt = \mathbb{E} \int_0^T \langle \eta(s), \int_s^T Y(t) dt \rangle ds. \end{aligned}$$

Thus (3.33) follows since $\eta(\cdot)$ is arbitrary. \square

Different from Theorem 3.6, in the above, we do not need the monotonicity of $t \mapsto \psi(t)$, and the conclusion (3.33) is weaker than (3.21).

Having the above result, we are able to get a comparison theorem for the following nonlinear BSVIEs ($i = 0, 1$)

$$Y^i(t) = \psi^i(t) + \int_t^T \left(h^i(t, s, Y^i(s)) + C(t)Z^i(s, t) \right) ds - \int_t^T Z^i(t, s) dW(s), \quad t \in [0, T]. \quad (3.35)$$

More precisely, the following theorem holds.

Theorem 3.10. Let $g^i : \Delta \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ satisfy (BV2) and the following hold

$$g^i(t, s, y, \zeta) = h^i(t, s, y) + C(t)\zeta, \quad (t, s, y, \zeta) \in \Delta \times \mathbb{R}^n \times \mathbb{R}^n, \quad (3.36)$$

for some $h^i : \Delta \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ and $C(\cdot) \in L^\infty_{\mathbb{F}}(0, T; \mathbb{R}^{n \times n})$. Moreover,

$$h^1(t, s, y) - h^0(t, s, y) \geq h^1(\tau, s, y) - h^0(\tau, s, y) \geq 0, \quad \forall y \in \mathbb{R}^n, \quad 0 \leq t \leq \tau \leq s \leq T, \text{ a.s.}, \quad (3.37)$$

and for either $i = 0$ or $i = 1$, $y \mapsto h^i(t, s, y)$ is differentiable with

$$h_y^i(t, s, y) \in \mathbb{R}^{n \times n}_+, \quad h_y^i(t, s, y) - h_y^i(\tau, s, y) \in \mathbb{R}^{n \times n}_+, \quad 0 \leq t \leq \tau \leq s \leq T, \quad y \in \mathbb{R}^n, \text{ a.s.} \quad (3.38)$$

Then for any $\psi^i(\cdot) \in C_{\mathcal{F}_T}([0, T]; L^2(\Omega; \mathbb{R}^n))$ with

$$\psi^1(t) - \psi^0(t) \geq \psi^1(\tau) - \psi^0(\tau) \geq 0, \quad 0 \leq t \leq \tau \leq T, \text{ a.s.}, \quad (3.39)$$

the corresponding adapted solutions $(Y^i(\cdot), Z^i(\cdot, \cdot))$ of BSVIEs (3.26) satisfy

$$\mathbb{E}_t \int_t^T Y^1(s) ds \geq \mathbb{E}_t \int_t^T Y^0(s) ds, \quad t \in [0, T], \text{ a.s.} \quad (3.40)$$

Proof. Observe the following:

$$\begin{aligned} Y^1(t) - Y^0(t) &= \psi^1(t) - \psi^0(t) + \int_t^T \left[h^1(t, s, Y^1(s)) - h^0(t, s, Y^1(s)) \right] ds \\ &\quad + \int_t^T \left[A(t, s) \left(Y^1(s) - Y^0(s) \right) + C(t) \left(Z^1(s, t) - Z^0(s, t) \right) \right] ds \\ &\quad - \int_t^T \left(Z^1(t, s) - Z^0(t, s) \right) dW(s), \end{aligned}$$

where

$$A(t, s) = \int_0^1 h_y^0(t, s, Y^0(s) + \beta[Y^1(s) - Y^0(s)]) d\beta.$$

Then under our conditions, we have the comparison (3.40). \square

3.3 Other type solutions to BSVIEs.

We now look at the following general BSVIE:

$$Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t)) ds - \int_t^T Z(t, s) dW(s), \quad t \in [0, T]. \quad (3.41)$$

According to [25], due to the appearance of $Z(s, t)$, there are infinite adapted solutions for (3.41), and under proper conditions, (3.41) admits a unique adapted M-solution $(Y(\cdot), Z(\cdot, \cdot))$. On the other hand, it is possible to define other types of solutions.

Recall that in the mean-variance problem, the precommitted solution is widely used when the objective function is

$$\mathbb{E}X(T) - \frac{\gamma}{2} \mathbb{E} \left(X(T) - \mathbb{E}X(T) \right)^2,$$

with $X(T)$ being the terminal wealth. Recently people started to study the dynamic version of

$$\mathbb{E}_t X(T) - \frac{\gamma}{2} \mathbb{E}_t \left[X(T) - \mathbb{E}_t X(T) \right]^2, \quad t \in [0, T],$$

and proposed the time consistent solution, see for example [3]. On the other hand, mean-field BSDE of

$$Y(t) = \xi + \int_t^T g(s, Y(s), Z(s), \mathbb{E}Y(s), \mathbb{E}Z(s)) ds - \int_t^T Z(s) dW(s), \quad t \in [0, T], \quad (3.42)$$

was introduced and studied in [4]. A dynamic version of (3.42) should be the following:

$$Y(t) = \psi(t) + \int_t^T g(s, Y(s), Z(t, s), \mathbb{E}_t Y(s), \mathbb{E}_t Z(t, s)) ds - \int_t^T Z(t, s) dW(s), \quad t \in [0, T]. \quad (3.43)$$

Inspiring by the above, we introduce the following definition.

Definition 3.11. Let $h : \Delta \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ satisfy (BV1). Moreover, $t \mapsto h(t, s, y, z)$ is \mathcal{F}_t -measurable for given $(s, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$. A pair $(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^2[0, T]$ is called a *conditional h-solution* for BSVIE (3.41) if (3.41) is satisfied in the Itô sense and

$$Z(s, t) = h(t, s, \mathbb{E}_t Y(s), \mathbb{E}_t Z(t, s)), \quad (t, s) \in \Delta, \text{ a.s.} \quad (3.44)$$

It is clear that for BSVIE (3.41), if we are talking about conditional h -solution, it amounts to studying (the usual) adapted solution for BSVIE (3.43) of mean-field type. By using a similar method in [25] or [22], we can establish the existence and uniqueness of adapted solution to (3.43). Then following the ideas contained in the previous subsections, we are able to discuss comparison of adapted solutions for such kind of equations. We prefer not to get into details here.

4 Concluding Remarks.

For BSIVs of form

$$Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s))ds - \int_t^T Z(t, s)dW(s), \quad t \in [0, T],$$

we have established a general comparison theorem (Theorem 3.2) for the adapted solutions when the tube

$$\left\{ [g^0(t, s, y, z), g^1(t, s, y, z)] \mid (t, s, y, z) \in \Delta \times \mathbb{R}^n \times \mathbb{R}^n \right\}$$

admits a selection $\bar{g}(t, s, y, z)$ which is nondecreasing in y , plus some additional conditions. Examples 3.3, 3.4, 3.5, and 3.7 tell us that when the above condition is not assumed, the situation becomes very complicated. At the moment, if the above monotonicity condition is not assumed, we can only prove a comparison theorem for the following restricted form of BSVIEs (see Theorem 3.6):

$$Y(t) = \psi(t) + \int_t^T \left(h(t, s, Y(s)) + B(s)Z(t, s) \right) ds - \int_t^T Z(t, s)dW(s), \quad t \in [0, T].$$

Further, if the generator depends on $Z(s, t)$, we need to use duality principle to prove a proper comparison theorem. Due to this, at the moment, the BSVIEs that we can treat is the following type:

$$Y(t) = \psi(t) + \int_t^T \left(h(t, s, Y(s)) + C(t)Z(s, t) \right) ds - \int_t^T Z(t, s)dW(s), \quad t \in [0, T].$$

Moreover, if $(Y^i(\cdot), Z^i(\cdot, \cdot))$ ($i = 0, 1$) are adapted M-solutions to the BSVIEs of the above form, instead of

$$Y^0(t) \leq Y^1(t), \quad t \in [0, T], \text{ a.s. ,}$$

(under suitable conditions, see Theorem 3.11), we only have a weaker form of comparison:

$$\mathbb{E}_t \left[\int_t^T Y^0(s)ds \right] \leq \mathbb{E}_t \left[\int_t^T Y^1(s)ds \right], \quad t \in [0, T], \text{ a.s.}$$

Theorems 3.2, 3.6, and 3.10 correct the relevant result presented in [23, 24]. Finally, the problem of comparison for the adapted M-solutions to the following general type BSVIEs:

$$Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t))ds - \int_t^T Z(t, s)dW(s), \quad t \in [0, T],$$

is widely open at the moment. We hope that some further results could be addressed in our future publications.

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